Mathematical Thinking in Movement

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Mathematical thinking in movement

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Introduction

This dissertation is first and foremost one among the potential paths throughout the last three years (of research), in which I have been carrying out a research project in mathematics education, attending a Ph.D. program in Pure and Applied Mathematics at the University of Torino. The line of research that I have pursued aims at studying the role of *movement* in mathematics and *in/for* mathematics learning. Since my very first engagement with research, I have been fascinated by readings coming from fields different from mathematics education, such as philosophy or anthropology, which attempt to address movement as an overarching concept that permeates the way in which we come to understand the world. Even though these sources might not be directly quoted or explicitly used in the present work, they have all contributed to particularly fuel my way of dealing with mathematics (and especially with mathematics teaching and learning) as something which might be better investigated if approached with extreme *sensitivity* to the ever-changing nature of life and interactions.

Inside this work, and in my whole research project as well, movement is relevant in more than one direction. I will now summarise these directions very briefly. First, the concept of movement informs theoretical concerns about mathematical concepts. Then, it characterises in fundamental ways the significance of bodily movements in mathematical practice. Finally, it grounds the specific mathematical activities that are discussed in this work.

Concerning the first point, I assume a vision of concepts as mobile in nature. In particular, the mobile dimension of mathematical concepts has to be ascribed to their *virtuality* (or potentiality). In my understanding of it, recognizing the virtuality of mathematical concepts means to trouble the commonly perceived 'definite image of abstract mathematics' and to look at the concept as something blurred, elastic and open to mobility. I also draw consistently on lines of research in mathematics education that study the role of the body and embodiment in mathematical practice, and that bring forth the necessity of deepening this matter from a non-dualistic perspective on knowledge. In my research, I pursue this line of flight to draw attention to the ways in which bodily movement and thinking are *contiguous* and push each other forward in mathematics. Lastly, my research project has involved the design and realisation of three classroom-based interventions in which a graphing motion technology (WiiGraph) was used for approaching the concept of function. The activities allowed to some extent to recover the epistemological roots for the concept, which are concerned with problems of motion (as widely pointed out in the literature). In addition, the implication of ample bodily movements in the activities to produce certain collectively shared mathematical representations also demanded bodily movement in fundamental ways. This further spurred an interest in investigating the phenomenological and *qualitative* dimensions of movement.

All the aforementioned aspects are deepened inside the dissertation, throughout theoretical and methodological sections, as well as through the analysis of collected data and its discussion.

In what follows, I will first give the reader a flavour of the structure and contents of the thesis, along with a clarification of the primary stylistic choices that have been made in assembling the present work. After this section, I will continue talking about the immersive role played by movement in the research process by discussing a short episode from a previous study, for how it opened up new questions and horizons, coming to be a novel beginning for my research.

Thesis structure and outline

This thesis is structured with main Chapters and additional Intermezzos.

An Intermezzo is a short chapter that is thought of as an element that creates a passage or smooths the transition between main chapters. In music, intermezzos have the peculiar characteristic of relating different movements inside a composition. In my work, an Intermezzo aims at presenting issues that would weigh the writing down inside a chapter, but that still constitute noteworthy pieces of the main discourse as a whole. Therefore, this choice resonates both with the function of an intermezzo, and its etymology ("being in the middle").

A first Intermezzo following this introduction is devoted to capture some important elements that constitute the theoretical background of my study, with particular focus on the *inclusive materialism* of Elizabeth de Freitas and Nathalie Sinclair (de Freitas & Sinclair, 2014). I also centre on the role that this perspective has had for the nourishment of my research process.

Chapters 1, 2 and 3 contain the theoretical commitments of the research.

Chapter 1 presents a literature overview of the studies that have tackled the concept of function as a crucial concept in mathematics education and that I see as meaningful for the present work. To better situate the discussion, I first propose a critical argument that digs into diverse theoretical views on concepts.

Chapter 2 is committed to unfold the philosophical concept of the virtual and its relationship with mathematics and mathematical concepts. The initial part of the chapter moves from Manuel DeLanda's reading of the work of Gilles Deleuze and his use of the mathematical metaphor of manifold for discussing the concept of virtuality (DeLanda, 2002). In the second part, I characterise some features of the virtual (or potential) according to my reading of Gilles Châtelet (1993/2000) and by drawing from scholars, who examine and use his work from the standpoint of philosophy, mathematics and mathematics education.

A second Intermezzo further delves into my interest in movement from the perspective of bodily movement, contextualising it inside recent literature in mathematics education that has focussed on the role of movement in mathematical activities and has investigated the relationships between movement and thinking.

Chapter 3 unfolds the concept of movement from the standpoint of phenomenology as discussed by Maxine Sheets-Johnstone (2009, 2010, 2011, 2014, 2016), in order to develop a discourse on movement that integrates cognitive, phenomenological and affective insights. This operation allows us to open up a perspective in which the processes of moving and thinking are integrated and coherently sustain each other.

The third Intermezzo closes the theoretical chapters through illuminating the interconnections among them.

Chapters 4 and 5 present the methods of the study. In particular, **Chapter 4** discusses the theory of design and teaching experiments and contextualises the interventions that have been planned and carried out in the classroom. Next, it offers an overview of the main design principles and of the longitudinal study. **Chapter 5** focuses on the research methods, concerning data collection and analysis. A brief discussion of the quality of the results will also highlight the ethical concerns of the qualitative inquiry I pursued for the present study.

In **Chapter 6** the research questions of the study are formulated. These questions interrogate the role of movement in mathematical practice in the particular context of using WiiGraph inside the mathematics classroom as a way to introduce the concept of function. Together with the theoretical commitments exploited in Chapters 1 to 3, they guide the analysis of the selected episodes, which are presented and discussed throughout the chapter.

Chapter 7 closes the main *corpus* of the dissertation by tracing the conclusions of the study, in an effort of answering the research questions and discussing open issues for future research.

Lastly, two appendices contain additional material for the reader. **Appendix A** presents WiiGraph, the software that has been used in this research study for our graphing motion activities. This technical description complements the methodological aspects already addressed in Chapter 4. **Appendix B** contains the original worksheets with written tasks, the questionnaires and the final tests, which have been faced by the classes engaged in the longitudinal study.

The following section is dedicated to shed light on an intermediate stage of my research work, as it initially spurred my interests. There are two main and interrelated reasons why this is a crucial step. On the one hand, I believe that it might be useful for the reader to gradually approach this research, by seeing connections with my previous work and tracing the tinkering process that shaped the study. On the other hand, research, as a process, is never fully exhausted in its own terms and is not a technical operation, but a way of living curiously with care for, and attention to, the world we inhabit[1]. Therefore, below as well as inside each Intermezzo, I will spend few words to touch on connections with some papers I have worked on during these three years. This will also help me put forward specific lines of thought that were crucial in my research and tell the reader little as well as momentous events that conditioned, nourished and fostered my research process.

Starting points and beginnings

A starting point for developing research interest for my Ph.D. project was my master's thesis, which was concerned with a research study in mathematics education. In that work I focussed on the use of some devices related to the Nintendo Wii game console inside the mathematics classroom as tools for the didactics of mathematics. In the research project, we designed, carried out and then analysed a teaching experiment that consisted of ten weekly meetings and involved a class of grade 9 students in Italy, in which all the activities employed the tools with specific computer software. One technology in use was WiiGraph, an interactive graphing motion application that leverages two Wii Remotes (or 'Wiimotes', the remote controllers for the Wii console) and that was developed at the Center for Research in Mathematics and Science Education of San Diego State University (CA) by Ricardo Nemirovsky and his colleagues. Among other things (see Appendix A), the software allows for graphing motion by capturing the positions over time of two users when they hold the controllers and move in front of a sensor bar, which is the origin for collecting the controllers' distances. As the users move in space, these distances are displayed on the screen as the two graphs of position vs time. The main interest of my master research was to investigate the role of proprioceptive and kinaesthetic engagement with the specific technology in mathematical practice for the introduction of functions via a graphical approach. Mathematics educators are increasingly interested in studying the role of the bodily interaction and how it partakes in the cognitive realm of mathematical understanding (see also Intermezzo: Movement). Drawing on this fruitful line of research and trying to develop an innovative curriculum proposal for the study of functions at upper secondary school, the research was considerably centred on the use of WiiGraph and the exploration of many of its modalities.

It is as if there were three people

In particular, in one section of my previous work, using mainly the perspective on mathematical instruments offered by Nemirovsky et al. (2013), I was investigating the ways in which the students were gaining fluency with WiiGraph while working in groups on the sum of two graphs (see Figure i). The students had explored the modality in which the software creates a third graph (additional with respect to the two graphs representing the position of each user), which is the sum graph. This graph captures, instant by instant, the sum of the two distances of the two remotes.

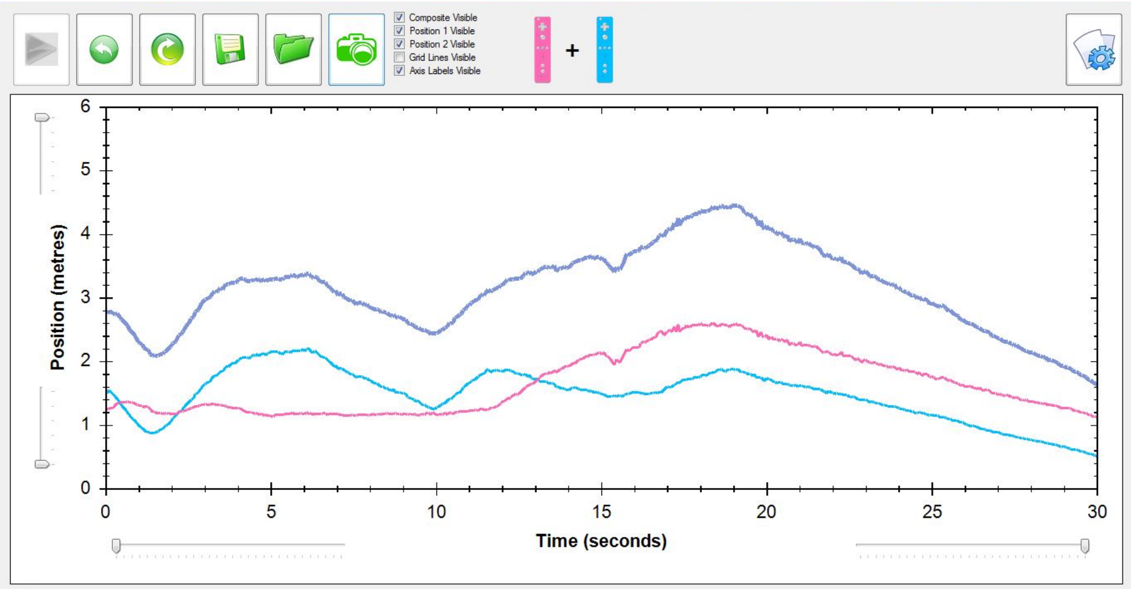

Figure i. a+b modality in WiiGraph (sum graph in dark blue; a and b in pink and light blue)

I present here an episode from the group work of three students, who were facing a written activity about the sum. These students had used for the first time the software in the sum modality, in the context of a collective discussion, during the session that preceded the episode. In that day, a couple of students made some trials, with exploratory experiments aimed at investigating what was the third graph appearing on the screen. After the classroom discussion, in which the students had first guessed the nature of the third line as the mean of the other two lines and then further investigated it as the sum, the students were asked to work in groups on the written task.

The first point of the worksheet required the students to imagine a situation in which they have to explain how the "*a*+*b* modality" works and its meaning to a friend with no experience in the use of the software. The three students, who are of interest in our discourse, are Luisa, Alessandro and Massimiliano. Massimiliano had been absent the day of the first activity, in which therefore he had not participated, while the other students had been challenged with experiments about the sum and involved in the associated classroom discussions. I focus here on an initial brief moment of the new group work (in the next day), in which Alessandro and Luisa explain to Massimiliano the meaning of the sum, after the group read the written task. The episode is presented with more details and expanded in the analysis with respect to prior investigations, as a way of connecting discourse to my previous work and to open up new questions and horizons for the current research.

The three students are sitting around a table, each on one side: Luisa sits next to Alessandro and in front of Massimiliano. Since the beginning, Alessandro and Luisa talk to Massimiliano by sharing ideas, completing each other's sentences, in a shared effort of explanation. They frequently gaze at each other looking for help and seeking for agreement, smiling when embarrassed or excited about what they are saying. Luisa begins telling that "First of all, the graph is the sum of the other two", and Alessandro adds "of the movements". Then, when Luisa looks down and says "So…" but stop talking and hesitates, he continues "the sum of the two graphs". We see that the two students are looking for ways of explaining something which is familiar to them, and which occurred quite naturally. Nevertheless, the sum of two graphs is not something trivial to define, and they struggle to capture it with words. At this point, the short dialogue develops as follows (L= Luisa; A= Alessandro; M= Massimiliano):

1. L:	So, there are two people,	*Looks down, keeps the elbows on the table and the hands lifted*
	that their graph, that is,	*looks behind Massimiliano, probably gazing the interaction space*
	each of them	*shifts the pen from left to right hand*
	performs a movement,	*with the pen in her right hand mimes some bumps in the air in front of her, gazes at Massimiliano*
	which is on the graph and,	*gazes for a moment and points with the pen to the graph area of WiiGraph, then gazes left to right and again at the interaction space, hesitating*
	that is, the graph	*mimes again the bumps in the air with her right hand, smiles, holds the pen with the left hand* (Figure ii.a)*, looks rapidly at Alessandro*
	is the sum of these movements of these two people,	*rotates the pen in the air* *stops, looks again at Alessandro, smiles to him, raises her eyebrows*
	and so	*looks down at the piece of paper on the table, lays the arms down on the table*
2. A:	It is as if there were, so, that is,	*Stops looking at Luisa and looks down, rolls up his sleeves*
	it is as if, say, there were three people,	*smiles as if excited, makes three with right hand, looks at the camera*
	that is, there are two people,	*turns to his left and points to the interaction space, where the people should be* (Figure ii.b)

	who perform two movements, and it is as, that is.	*mimes the two people moving with his two open right hand fingers slightly moving back and forth in the air, gazes at the interaction space*
		looks back at the interaction space
	If they stay, one at 1 [feet] and one at 2 [feet],	*Turns towards the researcher, mimes a quick movement in front of him, with his right hand moving a little forward in front of his torso*; Figure ii.c)
	it is as if there really was a third person, who moves at 3 [feet].	*turns again towards the interaction space*
	It is a sum, that is, the typical *a* plus *b* equal to *c*	*Looks at Massimiliano*
3. M:	Ah, yes	*Nods*
4. L:	As if there was a *c*	*Looks at Massimiliano*
5. L:	Right. That is, there the movement is that of *c*	*Repeats the previous bump gesture in the air with her right hand*

(a) (b) (c)

Figure ii. (a) Luisa: "each of them performs a movement"; (b) Alessandro: "that is, there are two people"; (c) Alessandro: "If they stay, one at 1 and one at 2"

Discussion

In this 1-minute segment extracted from the initial stage of the video of the three students' group work, we see how they struggle with finding a description and an exhaustive explanation for the graph of the sum. At first, Luisa recalls some issues that are significant in the modelling situation, namely that there are two people moving (when using WiiGraph) and that each of them creates a graph on the screen through movement. The sum graph is initially described by Luisa as "the sum of these movements of these two people" ([1]). Movements and graphs are fused through her gestures that actualise the graphs created by the people's movements. Luisa gazes all around, at his mates, especially Alessandro, whom she seems to be looking at for support, but also at the interaction space and the graph area where the graphs were shown during the previous collective discussions. When Luisa stops talking and looks at him hesitantly, Alessandro engages with enthusiasm in the discussion. He smiles to the camera when he says: "it is as if, say, there were three people" ([2]). In Alessandro's thought experiment, there are two people moving, as Luisa has already mentioned, but, additionally, it is "as if" there was a third person who moves in space, whose movements depend on the other two people's movements. Therefore, the movement of the third person is somehow constrained to the movements that are through of as already occurring in the interaction space, and the sum graph precisely depicts the movement of such third person, as if she was really moving. Alessandro also evokes the symbolic relationship by saying "It is a sum, that is, the typical *a* plus *b* equal to *c*", which in turn implies that the third person should be represented by that new, imagined variable "*c*" ([4], [5]), reaching Massimiliano's verbal approval ([3]).

Luisa's vision of the sum brings forth an initial status of the third graph, that is, its appearance as homogeneous to the other two graphs on the screen. Both in gestures and words she expresses how the third graph depends on the two users' movements as well as on the other two graphs. What matters is "the sum of these two movements", which seems to be unclear or insufficient even for Luisa, as she gazes at Alessandro with quizzical expression. What kind of graph is the sum of two movements? The students are acquainted with that which the graph of a movement is, but the question now is: What does this new, third graph *become* for the students?

Alessandro's intervention introduces a fresh element in the discussion: he proposes a sort of thought experiment with the third graph, which is the object of inquiry, that asks for the presence of a third person. I should underline that the tool is not in use in this activity, nor is required for a third person to be involved in any experiment of motion in order to obtain a third graph, in this case the sum graph. In addition, the third person cannot move freely, rather has to move constrained to the other two people, as evoked by Alessandro. Essentially, the tool is reimagined in this though experiment, which operates to reconfigure the activity and the nature itself of the graph. Ascribing a third person to the third graph, Alessandro is in fact recalling once more that the third graph is homogeneous to the other two. Further, we can see that his experiment suggests that the third graph possesses very specific characteristics that it shares with the other graphs. First, we might say that Alessandro even considers the third graph as a function or, rather, as capturing spatio-temporal relationships. The third line's shape and movement have to be bound to the other two graphs as well as to the connected movements, but it is still a graph. We might be tempted to stress that the relationship is then "generalized" by saying that the use of the symbolic notation "a+b=c" makes room for extending the single example of Alessandro to the multiplicities of cases in which the sum is achievable.

Rather, what is intriguing the most is that the letter c serves the students *to give a name* to the third graph, and that its use collects the consensus of the three students. Luisa, in fact, agrees and uses the symbol again just to recall the third graph ("As if there was a c").

The third person/movement is therefore present because there is a third graph. It is not only the case that the technology allowed the students to encounter the sum of functions as a new function produced instant-by-instant by a standard numerical sum. Instead, the tool is reinvented through a reconfiguration of its peculiar ways of functioning, which brings about a new, dynamic vision of the sum. These aspects are also actualised in the written, when the students say: "The two people move in front of the sensor in the same way in which they moved the other times but, on the graph a third movement is represented, which is the sum of the first two". Attention is also drawn to the coordinated bond: "the first thing is that it's necessary to collaborate". This collaborative nature of movement captures the homogeneity of the sum (c) with the pink and blue lines, and the development of a symbolic understanding of the sum: beyond a+b=c, a=c-b and b=5-2 if a=2 and c=5 (Figure iii).

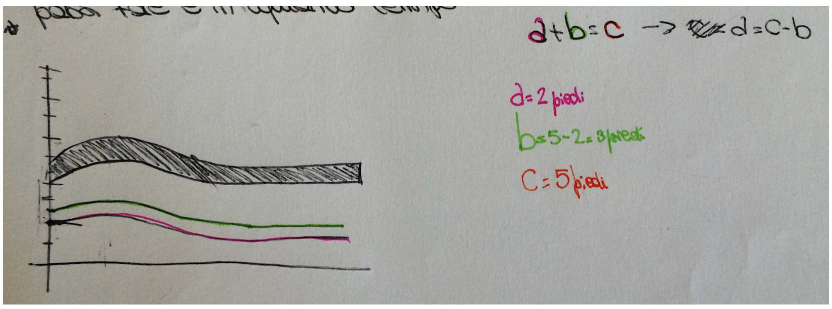

Figure iii. Part of the written protocol of the three students

New questions and horizons

The episode was a considerable source of thoughts and reflections, which made space for new horizons in my doctoral research. Besides the great interest in the potentialities of WiiGraph, which clearly was a relevant ingredient that I decided to preserve, there is much more. Not only the software fuels interesting and immersive thought experiments like that of Alessandro or creates new forms of diagramming and gesturing in the classroom (see Ferrari & Ferrara, 2018). But also, activities with the software like the one concerning the sum create new modes of navigating the concepts at play in the activity, like that of function.

In the brief example, we see the students working on a task that asks them to explain a specific graph modality that they have explored with WiiGraph. They are working hard with finding a satisfying way of defining what in their experience was simply a third graph appearing on the screen. This struggle seems to expand as Luisa gazes at Alessandro and then to contract at the point in which consensus is reached towards the naming of the third graph. We noticed how affective tones that circulate in the minute interactions of the students sustain the activity, developing through the heterogeneity of the interactions. We can grasp this subtle and pervasive circulation of affects by posing dedicate attention to the ways in which the bodies are caught in this movement, which is neither of one body or another.

Movements traverse multiple places, as some gestures and changing postures of the students occur over and around the surface of the table, and intermittently are directed towards the interaction space, and other gestures that actualise graphs are created in the shared space around the three bodies or projected towards the graphical space of the software. Moreover, we see how the gestural does not only represent but makes present and actualises that which is spoken about, like the graphs created by Luisa and the third person involved in Alessandro's thought experiment.

We also discussed how the third graph, which is at the core of the students' discussion, emerges from a symbolic relationship: the perceptuo-motor aspects of the activity merge with the imaginative and are entangled with the symbolic. The symbolic notation is introduced beyond explicit request, as a way for the students to look for and ultimately find consensus. It is also used to establish a relationship between the three people of Alessandro's argument, which I interpret as capturing the relational movements inside.

Therefore, we can see how the perceptuo-motor, the imaginative and the symbolic feed one another, and are only separable analytically, after the experience, while together they constitute the richness of the mathematical encounter of the students.

The episode thus brings forth the relational nature of graphs in WiiGraph and how the instrument can fuel thinking processes starting from a clear focus on movement. Broadening the view with which we look at the episode as a classroom event, the discussion would also enlighten mathematical activity as a creative act of disruption, which reconfigures what is known, what is perceived and what is imagined, through movement.

Movement is indeed the key element that guides my explorations towards and through this dissertation.

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[1] This sentence is my personal re-elaborated version of Tim Ingold's thought on the meaning and purpose of research, expressed during the speech he delivered at GAM (Galleria civica di Arte Moderna) in Turin ("Art, Science and the Meaning of Research", 28th March 2018).

Intermezzo: Inclusive materialism

A new framework in mathematics education

This intermezzo is devoted to capture some important elements that constitute a background for the theoretical framework of my research, with particular focus on the *inclusive materialism* of Elizabeth de Freitas and Nathalie Sinclair, a new perspective in mathematics education.

De Freitas and Sinclair (2014)'s book, "Mathematics and the Body" has challenged my way of thinking about mathematics, in the first place, as I will detail in this Intermezzo. The vision of mathematics as a discipline and the implication in and for education, particularly the perspective on concepts (see Chapter 1) is grounded in new materialist studies, which were an unusual theoretical scenario for mathematics education research until few years ago. The inclusive materialism is part of a trend within the field, namely the proliferation of research studies that look at the mathematics classroom events with a post-human sensitivity to human and non-human bodies (materials), studies which were more common in the social sciences. Post humanist approaches seek to reconceive the human criticism to humanism, which claims that human nature is a universal state from which the human being emerges (autonomous, rational and capable of free will). Denying human exceptionalism, a posthuman position recognizes imperfectability and disunity within herself and understands the world through heterogeneous perspectives while seeking to maintain intellectual rigor and dedication to objective observation (Braidotti, 2016). Posthuman has an emergent ontology rather than a stable one, as provisional determinations and fluidity of relations inform the way in which we can understand the world.

Two main theoretical lines inform the inclusive materialism of de Freitas and Sinclair:

(1) the work of the physicist and philosopher Karen Barad, together with feminist studies, with particular focus on the notions of intra-action and entanglement, discussed within Barad's theory of agential realism (e.g., Barad, 2007); and

(2) Gilles Châtelet (1993/2000)'s vision of physico-mathematical concepts, which grounds the discussion in the ontology of mathematics as it will be offered in Chapter 1.

A common element between these two driving forces is the work of the philosopher Gilles Deleuze and his philosophy of difference. In particular, assemblage theory is the glue to reconceive dynamic and provisional relations as dispersed across heterogeneous bodies, attending to their provisionality and emergence, while also focussing on the political and affective forces that populate those interactions. As a key point, becoming (as a process) rather than being (as a state) can reverberate the indeterminacy and mobility that are characteristics of collective and individual assemblages. Concepts emerge out of "material-discursive boundary making practices" (Barad, 2007, p. 148), which are seen as the material arrangements of concept, student, tool and movement. Concepts are not purely abstract insofar they partake of the physical world. De Freitas and Sinclair adopt "a theory of matter that resists the binary divide between human agency and inert passive matter" (de Freitas & Sinclair, 2014, p. 39); they revitalize materials, which are often considered to be passive or inert, and re-animate concepts, which are often considered detemporalised, dissolving boundaries between bodies and concepts (Chorney, 2014). As a result of this shift, de Freitas and Sinclair (2014) turn up-side down the kind of question one might be tempted to pose in traditional learning context in mathematics education. Some examples among (possible) others. Instead of asking how an abstract, detached from reality, mathematical concept get to be incorporated or conceptualized by someone, they ask:

> "Does mathematical activity actually entail a remixing of matter and meaning in such way as to reconfigure the world?" […] "In what ways do more sophisticated mathematical diagrams continue to hail the embodied viewer?" (p. 15) […] "How do mathematical concepts change when they partake of this kind of activity?" (i.e. tapping, moving, …) (p. 150)

Affect and sensation permeate the assemblage, which comprises humans as well as non-human bodies (tools, surfaces, objects, …), therefore the center of will and intentionality is not condensed into the human body but gets to be dispersed within the assemblage. The authors' interests also rely in highlighting "microvisceral movement of mathematical activity, while also attending to the enduring political forces that operate through material assemblages" (p. 57). Thanks to a posthuman approach to sensation, de Freitas and Sinclair even look at perception not as the synthesis (rational judgement) of sensations like in the Kantian theory of perception, "but rather [as] a polyphonic process of modulation, a process by which new folds and inflections emerge in unstable material configurations" (p. 157). This speaks directly to a radically divergent way of thinking about embodiment, one that aims at emphasizing potentialities as well as indeterminacy that guide perceptions and bodily movements, posture, gestures, and so on.

As these researchers acknowledge within the book,

> "In wanting to attend to the collective nature of these acts, and to the ways that non-human materials factor into the process, we were faced with challenging methodological choices. Embedded throughout the book are attempts to look at data differently and to reconsider what constitutes research data more generally. This kind of experimental work is important because of the way it forces us as researchers to reckon with the radical limitations of our research methods." (p. 12)

Lines of research

For the sake of clarity in relation to the aim of this Intermezzo, it is worthwhile to name some central issues that are at the core of inclusive materialism, especially those which were central in nourishing my research process. In particular, this nourishment has occurred along three main directions:

(1) troubling common assumptions about mathematical concepts and mathematical doing;

(2) shifting the focus in the ways in which the body has been traditionally conceptualized in mathematical practice;

(3) looking for ways of speaking and methodologies capable of sustaining this vision of mathematical encounters.

Therefore, I chose to take inclusive materialism as my *a priori* background in mathematics education, as it was influential in my research. My way of thinking with theory is not that of putting at work categories or (even dynamic) interactions among them, but it is more about generating new questions and meanings through difference. This will be clarified throughout Chapters 2 to 5 as I will discuss the theoretical framework (Chapters 2 and 3) and the methodological concerns of this research (see Chapters 4 and 5).

The search for methodological resources that could assist the theoretical approach (and new ways of speaking) has particularly stimulated my research process. This is one of the reasons why in this dissertation I explore ways of tapping into perception via a detailed analysis of movement, opening room for a phenomenological account of movement as a way of grasping its enigmatic nature. Movement will assume many nuances as the discourse progresses, ranging from bodily movement and the flow of sensations (the ways in which bodies are reconfigured in the learning encounters) to the becoming of the concept and of the mathematical subject in the learning assemblage. Many of these nuances have their origins precisely in my understanding of inclusive materialism, as it will be discussed in the following.

In particular, the concepts of assemblage, agency and becoming are key ideas of inclusive materialism, which have inspired the theoretical interests in my Ph.D. research project and related research work. Here, I will shortly depart from the initial discourse to leave the reader with a sense of how these three theoretical notions have been further studied and developed.

First, the idea of *learning assemblage* (de Freitas, Ferrara, & Ferrari, 2017) recalls assemblage theory as it is discussed within the social sciences by DeLanda (2006), who gives a reading of the Deleuzian concept of assemblage[1]. In that work, we investigate how a sense of locomotion and coordinated movement is entailed when students use Wii graphing technology to explore mathematical relationships. The concept of learning assemblage is used to discuss how coordination and agreement inform movement in technology-based activities, in terms of the heterogeneous and provisional relationships that emerge between moving parts (within the assembling of students and technology). In particular, a learning assemblage gets to be assembled through forces of affect (and not simply through mechanistic coordination), so that the study eventually evolved into the exploration of *sympathy* in collaborative mathematical tasks, as a way of studying the flowing of affect within the heterogeneous bodily coordination involved in the creation of a circle (de Freitas, Ferrara, & Ferrari, 2018). In studying the sympathetic bonds that sustain collaborative mathematical activities, the mathematical concept matters. The concept is implicated in the activity in different ways, also in terms of the affective forces at play in mathematical practices.

Another object of interest is the notion of *agency*[2], rethought as dispersed and distributed in the mathematics classroom, particularly among students, patterns and diagrams in the context of pattern generalization (Ferrara & Ferrari, 2017). Such theoretical understanding for the concept of agency informs a vision of mathematical activity, which tends to highlight a mobile ontology. In fact, whether we take a dynamic assemblage as unit of analysis, we are forced to re-think (1) which are the forces that are at play within any activity, and (2) how they get to be composed to sustain dynamic interactions. In the paper, we analysed the work of three girls dealing with a task that uses numerical and figural patterns. The task was designed to provide opportunities for students to reason about relations between variables and formulas. Tracking how agency was flowing across the assemblage of students-pattern-diagram allows us to highlight the agential power of the diagram, whose becoming in the discovery of mathematical relationships is speaking directly to the emergence of new mathematical meanings. The girls' bodily movements around the diagram, together with the task and the diagram itself, are in a constant process of becoming, so that the ontology of the pattern is shifting moment-to-moment in concert with the activity.

Lastly, attention to the concept of *becoming* informs a vision of movement that expands the solely (human) bodily movement and comprises movement, in a wider sense, as an emerging property of all kinds of mathematical activity.

Dwelling into the enigmatic nature of movement has prompted me to develop two main ideas (1) from the phenomenological perspective of Maxine Sheets-Johnstone (2011), the vision of "The Primacy of Movement", that is, movement as organising principle for a comprehensive understanding of animate beings, and (2) the mobility that has to be granted to mathematical concepts, in terms of their virtuality, as it is proposed and discussed in the work of Châtelet (1993/2000), "Les Enjeux du Mobile". These two theoretical underpinnings are at the core of my theoretical framework. The framework will be expanded in Chapters 2 and 3, while two Intermezzo will connect and diffract ideas emerging throughout Chapters 2 and 3.

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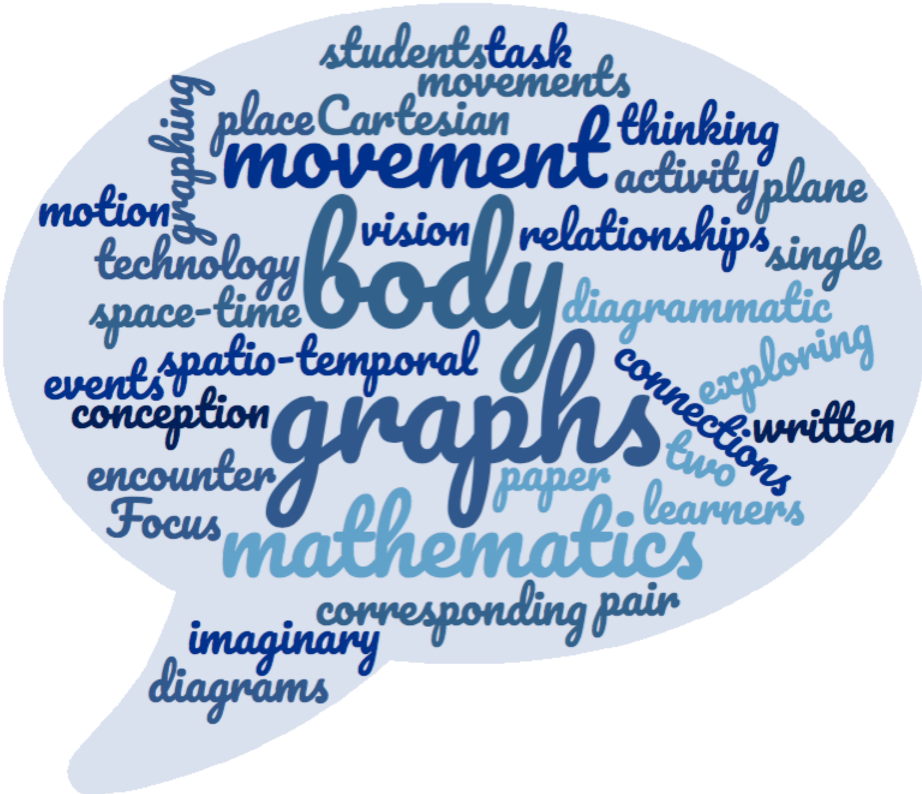
[1] An assemblage is a porous and complex body: it is an ambiguous concept that has its derivation from the work of Deleuze and Guattari (Deleuze & Guattari, 1987). In DeLanda's interpretation, an assemblage is a topological concept that invokes a structural generative process, along the dimensions of the temporal, the material, the relational and the perceptual (DeLanda, 2006). Particularly interesting is, say Marcus and Saka (2006), that "the time-space in which assemblage is imagined is inherently unstable and infused with movement and change" (p. 102). Within the assemblage, emphasis is on relations between movements, from which reality progressively emerges rather than be built or structured. Any more detailed description of

assemblages would bring us far apart from our discourse, but the reader would find many similarities with the concept of multiplicity, which will be unfolded in Chapter 2.

[2] The Stanford Encyclopedia of Philosophy broadly defines the term 'agency' as denoting the exercise or manifestation of the capacity to act by some being. Traditionally, agency is conceived as the result of the intentionality of the subject, who is supposed to possess the capability and will to act. Alternative conceptions of agency, like that of Bennet (2010), show how the fact that "bodies enhance their power in or as a heterogeneous assemblage" suggests for the concept of agency "that the efficacy or effectivity to which that term has traditionally referred becomes distributed across an ontologically heterogeneous field, rather than being a capacity localized in a human body or in a collective produced (only) by human efforts" (p. 23). Therefore, agency needs to be reconceived "as operating within the relations of an ever-changing assemblage, a force that flows across the encounters" between artefacts, hands, voice and other bodies (de Freitas & Sinclair, 2014, p. 33). Drawing on Rotman (2008), the body is no longer confined to the flesh borders of the individual person, but it must be conceived in terms of distributed agency across a network of interactions, the properties of which are constantly changing. For him, this entails a process of *becoming beside ourselves*, which captures the acentred sense of subjectivity that emerged in this century, and a network "I" that thinks of itself as permeated by other collectives and assemblages. According to this perspective, artefacts in the mathematics classroom, including the paper, the pencil, the compass, the digital tools and the diagrams, have some degree of agency. They participate in agential relationships with the user so that the user and the artefact mutually constitute each other through interaction (de Freitas & Sinclair, 2013). This post-humanist understanding of agency implies that subjects are constituted as dynamic assemblages and that the mathematical subject comes into being through these material and social encounters. The body of the concept also matters in agential terms. The mathematical concepts (e.g., of cube or circle) are part of the assemblages and engage in a process of becoming that binds them to the actions of the students/mathematicians.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1

Literature review

Any research work is populated by questions or dilemmas that make the whole process alive to the researcher and that characterise the way in which it is crystallized into the final, written product.

While writing a literature review in this chapter, to adequately insert the issues that are addressed in this dissertation within the research field of mathematics education, I found myself caught in a dilemma: How am I going to write about the concept of function, if I do not clarify to myself and the reader which perspective I am taking on concepts?

The following sections are the result of the path that I have created throughout facing the dilemma, seeking answers to this question, delving into theoretical views and approaches that have accompanied my own research work. A sufficiently open panorama will help us see the relevance of the original dilemma in such discourse.

1.1 Premise: On the concept of function

I share with Thompson and Carlson (2017) the idea that something like the "concept of function" does not exist. This sentence may sound provocative and indeed one might argue that there is wide research on the concept of function, especially in the field of mathematics education. How could not the concept of function not exist, if that something lies at the core of so many scholars' discussion and is so crucial for the mathematics of the last five centuries? The point that Thompson and Carlson (2017) make is that there are different ways of envisioning what a function is. They write:

> "A student's conception of function will not be as developed as that held by a mathematician, and a mathematician's conception of function may not include detailed information that a math education researcher has about how students' function understanding develops" (p. 421).

Far from being just a question of much or less detail in the information possessed by an individual, Thompson and Carlson describe in depth different studies that have tackled the issue by specifying various meanings and ways of thinking, which they envision a person having a concept of function holds (cf. §1.5).

One might ask, then, which are the features or components that constitute the concept of function 'itself': Is the concept of function a kind of collection of all those ways of thinking and all those meanings?

One possible answer comes from Anna Sfard's (2008) theory of commognition, where she argues that mathematics is some kind of discourse. Specifically, Sfard discusses the nature of mathematical objects as they arise from mathematical discourse and in relation to what she calls "realization trees": a realization tree is a personal construct, highly situated, that establishes hierarchical and ontological relationships between signifiers and the realizations of those signifiers.

Each branch in a realization tree connects different but somehow related facets that are constituted within a person's discourse (thinking) around, say, a parabola. In my understanding of it, the continuous graph of x^2 in the Cartesian plane, the equation $y = x^2$, a table of values for that function, a *dynagraph*[1] for the same function, and so on, might be elements in the realization tree of a parabola.

A tree of realizations is seen as a provisional but ordinated bundle of perceptually accessible realizations of a mathematical object, which is not complete but can illuminate the interrelations that exist within one's discourse on that object. Apparently, going back to Thompson and Carlson's quotation above, a mathematician's tree of realization about the mathematical object "function" might be wider and more complex than a student's tree. It might comprise a whole spectrum of branches that goes from graphical to symbolic expressions, including second-degree equations, but also conic sections, parabolic equations and geometry, and so forth, while the student's tree is probably limited to few branches.

But, then, in which sense do these branches all resonate with the concept of function? How do they all contribute to what a function might be for an individual? We will leave those questions partially open here since answering them goes beyond the specific interest of the present chapter. However, posing the issues is relevant *per se*. On the one side, it suggests how intricated might be the unfolding of what a mathematical concept is. On the other side, the discussion itself brings to the fore that possible answers to these issues might have influence on education, whichever stance we take to frame our research.

Concerning our interest on function, Sfard (2008) also traces the historical development of the concept by showing how discourse on function aroused from the *saming*[2] of discourses around Descartes' work on algebraic formulas and Cartesian curves and around attempts of modelling physical processes (like it was with the problems of falling bodies and vibrating strings). Following her interpretation, we might say that discourses on function subsumed discourses on algebraic formulas, on curves and physical processes, and they all together concurred to the formation of the concept thanks to this process. In doing so, Sfard discusses how new definitions of function emerged as reactions to the impossibility of describing certain types of physical phenomena with a single formula, which "rather required what we now call a *split-domain* function" (p. 176, *emphasis in the original*).

Early definitions of function are those given by J. Bernoulli:

> "One calls a function of a variable a quantity composed in any manner whatever of this variable and of constants" (Bernoulli, 1718, quoted in Kleiner, 1989, p. 284)

and by L. Euler:

> "A function of a variable quantity in an analytical expression composed in any manner from that variable quantity and numbers or constant quantities." (Euler, 1748, quoted in Kleiner, 1989, p. 284)

Few years later, Euler proposed a different definition, writing that

> "If x […] denotes a variable quantity then all the quantities which depend on x in any manner whatever, or are determined by it, are called its functions." (Euler, 1755, quoted in Kleiner, 1989, p. 288)

New definitions like Euler's no longer contained reference to the symbolic (algebraic expressions or graphical representation): in Sfard's terms, these definitions became more inclusive and more objectified. Sfard (2008) also points out that "[T]his time, rather than being a mark on paper, function presented itself as a *disembodied abstract entity*, existing independently of its perceptually "accessible avatars"" (p. 176, *my emphasis*), since any reference to visual and symbolic realizations is lost in new definitions from that moment on. The move towards a more abstract and disembodied vision of function is then apparent with the work of Bourbaki, in which the definition of function[3] only used set theory to establish a correspondence between the independent and the dependent variable (this definition is today widely considered "The definition" of function and such appears in most of mathematics text for secondary education in Italy). Bourbaki's definition is subordinating all what before happened in the plane with curves and formulas to a certain type of relation that can be expressed somehow—not necessarily in an analytic way—between the elements of two given sets. Such understanding of function is mainly drawn from the two definitions that were independently formulated by Lobachevsky and Dirichlet, which insist on the uniqueness of the second element associated to the first one, and the fundamental contribution of Dedekind, who defines a function as a single-valued mapping between any two sets[4].

The modern concept of function, the Dirichlet-Bourbaki concept subsumed the influence of the logician's turn towards a formalized mathematics, and a deep interest of mathematicians in building the Foundations of Mathematics. David Hilbert's work on the axiomatization of mathematics and his dream for the accomplishment of a consistent theory, led to a definition of function in terms of the existence of an object and used quantifiers as means for establishing its existence.

This brief excursus brings forth how the definition for the concept of function changed through the course of history. Sfard's commitment to seeing mathematics as a discourse and mathematical objects as discursive constructions is consonant with the choice of focussing on definitions, which are formal, definit(iv)e, verbal descriptions that crystallize mathematical discourse over the course of history. However, while the commognitive theory tries to capture the process of formation of concepts, it does not really keep trace of material activity and its importance in (the real process of) doing mathematics. This is the main reason why, even though I recognize the huge relevance of the commognitive approach, which is indeed a well-established and widely used perspective in mathematics education, in this dissertation I will choose to an alternative path to dwell on the concept of function. There are two main points at the ground of such a choice. (1) First, from an empirical point of view, this work wants to focus on and explore the concept of function in the ways that it is encountered by students in the context of graphing motion experiences. The activity of the students fundamentally embraces bodily, gestural, material, spatial, temporal dimensions of doing mathematics. Putting aside these dimensions in favour of words and discourse seems to be too constraining. (2) In addition, from a theoretical point of view, I will draw on some studies to show the reader that whether we assume the virtuality of mathematical concepts or we treat them as multiplicities rather than essences, whether we take concepts to be material arrangements (or devices) or we take mathematics to be a *practice*, we need to add other elements to the historical reconstruction[5]. In fact, we are not just interested in what discourses around (and endorsed narratives on) curves, graphs and formulas *say* on functions (as it would be in commognitive terms), but especially in *how material activity with and on functions captures and unfolds some instances of this concept*.

Taking into account such matter allows us to treat the development of the concept of function within a wider perspective, while capturing how it assumes the nuances not simply of a logical-discursive progression, but of a much more complex articulation and formation.

Of course, many questions arise from this shift of interest. For example: "What kind of *material* aspects should we consider?", or "In *which sense* they are significant?". But also: "If so, *how* do they contribute to our understanding of the concept of function?".

Substantially, the main point I want to make is that, insofar as we limit ourselves to consider concept definitions and stages in the process of objectification to unfold the whole story, we might lose some insights on the complexity of the process, especially in the mathematics classroom. Disruptions and deviations are part of the history of formation as logical improvements and progressions are. The chapter wants to offer the reader sort of insights about these issues.

We begin developing this argument touching on more general thoughts on mathematics and the ontology of mathematical concepts, which situate why and how our approach might be relevant.

1.2 On concepts

In a recent book, a wide group of scholars has been tackling the huge question of what a mathematical concept is, as a way of tapping into the ontology of mathematics from many vanguard theories in the humanities and post-humanities (de Freitas, Sinclair, & Coles, 2017). In so doing, they offer us ways of addressing an even broader question: What is mathematics? In a less recent work, Rotman (2006) argues that there are only three serious responses - mutually antagonistic and incompatible - to this last question. These responses encompass three different philosophical stances on the ontology of mathematical concepts: formalism, intuitionism, and Platonism. Briefly speaking: according to the three philosophical stances, mathematics might be respectively seen as "a meaningless game, a subjective construction, and a source of objective truth" (p. 101). According to Rotman, each of these stances seems in some sense inappropriate to adequately take into account the practice of mathematics, and to ground a coherent account of how mathematical practice creates mathematical knowledge.

The author proposes a semiotics of mathematics that relies on the idea of a plural "I" (composed by Mathematician, Person and Agent) to which the "truths" of mathematics have to be attributed. In the context of numbers, he shows that mathematical thought and scribbling enter into each other and that mathematical language creates as well as talks about its worlds of objects. As a consequence, he poses that all the theories that state separateness of objects from their descriptions are unable to capture the mathematizing process, Platonism in the first place.

I believe that, within the context of education research, de Freitas et al.'s (2017) book consists of a potential follow-up to Rotman's perspective, by offering an interdisciplinary collection of chapters that dwell on the nature of concepts by drawing on recent developments in post-constructivist learning theories. In the introduction, they state that such theories

> "have shown how concepts are performed, enacted or produces in gestures and other material activities (B. Davis, 2008; Hall & Nemirovsky, 2011; Radford, 2003; Roth, 2010). This new theoretical shift draws attention to how concepts are formed in the activity itself rather than in the rational cognitive act of synthetizing (Brown, 2011; Tall, 2011). This work reflects a paradigmatic shift in learning theory, driven in large part by offshoots of contemporary

> phenomenology, better to address the role of the body in coming to know mathematics." (de Freitas et al., 2017, p. 5)

This positioning constitutes a consistent move towards overcoming theories that insist on separation and continue to renew dichotomies. In my interpretation, it is also a socio-political move for mathematics as a discipline, and one of the reasons for which it matters here. In the following, we will draw on the theoretical stance offered by de Freitas and Sinclair (2017) on concepts as generative devices, as a fertile ground to develop our discourse.

It is worth noting that the vision that these researchers take of concepts arises from considerations on the role of the body in mathematics and their implications in mathematics education (de Freitas & Sinclair, 2014; Intermezzo: Inclusive materialism). Refusing Kantian-based epistemologies, which reduce knowledge to be imposed by external schemas, and basic assumptions of Platonic realisms, which reduce mathematical objects to be external and immutable entities, de Freitas and Sinclair give a fundamental contribute to new theories of embodiment and considerably disrupt common assumptions that in the last decade have characterised the approach to the study of the body in mathematics (education). Coherently with this vision, in their more recent work (de Freitas & Sinclair, 2017), these researchers have further developed a post-humanist approach to concept formation, reinforcing the idea that learning is about encountering the mobility and indeterminacy of concepts.

In particular, they write:

> "Trying to understand how seemingly abstract concepts become parts of body-assemblages does not simply involve locating the concrete sensori-motor activities that supposedly give rise to mathematical concepts as metaphors. When Lakoff and Núñez (2000) describe the container metaphor from which the mathematical idea of 'set' emerges, they treat concepts as metaphorical representations of the real world. If concepts are only metaphorical in relation to the 'real', then we are forced to wonder how and why that metaphorical relationship holds. Metaphoric relationships operate according to a double-standard ontology, where the mathematically abstract and the physically concrete are mutually detached and then reconnected through analogy or resemblance. Such an approach tends to treat concepts as representations rather than material arrangements, and it is this *material arrangement* in which we are most interested." (p. 76, *my emphasis*)

The notion of assemblage, taken up from Deleuzian philosophy, consistently informs such view on bodies: the collective social body, as unit of analysis, rather than the individual collective body, can support the vision of activity as complex entanglement of social and material. In particular, in a learning assemblage, mathematical concepts are taken to be *material arrangements*, which means that they do not simply exist in an *a priori* world but are materially implicated in a process of becoming. In other words, they are not already determined in their already fixed meaning, but continuously unfold during activity (Barad, 2007). De Freitas and Sinclair also draw on the work of Châtelet (1993/2000) to account for mathematical concepts as generative devices. Châtelet's perspective considers mathematical objects as physico-mathematical entities, which function in light of a tension between the real and the virtual, the experimental and the abstract, the physical and the mathematical. In other words, mathematical concepts are "material objects on and with which mathematicians perform thought experiments. These thought experiments are not the disembodied mental ruminations with which we typically associate mathematical thinking but, rather, gestural choreographies and exploratory diagramming" (Sinclair & de Freitas, 2014, p. 562). In particular, it is the virtuality of mathematical concepts, to put it simply, their openness to modifications and alterations[6], that grants concepts with their generative and mobile character. In line with these two theoretical underpinnings, the authors describe the nature of mathematical concepts through a bunch of meaningful propositions, as follows:

1. Concepts are not merely metaphors or representations;
2. Concepts are not mental constructs abstracted from the material world;
3. Concepts are vibrant and indeterminate, having one foot in the virtual and one in the actual;
4. Concepts operate as both logical and ontological devices;
5. There is no *a priori* logical ordering between mathematical concepts;
6. Concepts emerge from aesthetico-political acts.

Rather than discussing the previous propositions in detail, I propose the reader a playful example and then a discussion of the main points that, in my understanding, are crucial to be pursued within this view. Let us think of the apparently simple concept of circle. We might think of a circle not just as the locus of points equidistant from a given centre, but also as the material trace of an object that is subjected to two opposing forces. If thought of in the first way, the circle realises the possible (the given rule) and adheres to logical constraints inside it. The second way puts a light on the generative, mobile activity of forces that produce the circle and it does not adhere entirely to logical determinations (the idea of a prefixed circular shape) but makes the circle a dynamic concept (a point/object moving according to physical forces)[7]. It is exactly what Châtelet meant: both dimensions coexist in mathematics. Each of them brings forth a distinct trait of the circle as a concept; in particular, the second one engenders the circle as a mobile entity, which does not exist insofar as we put it in motion. This example is far from being exhaustive, but it wants to give the reader a first glance on the ways in which de Freitas and Sinclair propose to rethink the ontology of mathematical concepts.

Notably, the perspective offers a way to think of the mathematical activity (at all levels, kindergarten or university, in all learning context, from formal to informal) as inherently working on mathematical concepts themselves. In other words, if we forget for one moment the idea that the practice of mathematics only concerns abstract and disembodied objects living somewhere else, we could grasp the immense power of this different position, despite initial possible hesitations in considering mathematical concepts as material. If we do so, we can grasp the ontology of mathematics as always mobile, shifting, provisional. Concepts are not reality-detached entities but are implicated in the moving hands and the speaking mouth and traversed by streams of affect as it flows within and throughout the (learning) assemblage.

The indeterminacy of mathematical concepts creates the possibility for the new to emerge, for a mathematical concept is not already determined prior to our engagement with it. Concepts are not transcendent universal ideals, but "operative and highly flexible arrangements or apparatus *in this world*" (de Freitas & Sinclair, 2017, p. 87, *emphasis in the original*). For mathematicians, as well as for students encountering concepts that are new to them, the new arise from direct engagement with concepts and not from reality-detached abstraction we can build of a mathematical object. The profile that de Freitas and Sinclair trace for concepts resonates with the idea of multiplicity, in the sense of Deleuzian philosophy (e.g. DeLanda, 2002), that is inevitably counterpoised to that of essence (see Chapter 2). Thinking of mathematical concepts in terms of multiplicities grants them a fundamental mobility, and openness to movement. Mathematical concepts have to be granted degrees of mobility in the first place.

In this perspective, concepts are treated as ontogenetic devices, which need to be reanimated into the curriculum (see also §2.3). Concerning this point, we will investigate in the following a proposal to reanimate the concept of function in the context of a longitudinal study (cf. Chapter 4). Moreover, concepts are not something that human beings acquire or have, but they grow in the making, they are in becoming within assemblages and subject to formation and deformation as is with matter.

Borrowing fundamental ideas on concepts from de Freitas and Sinclair (2017) and prompted by one of the main issues of "What is a mathematical concept?" book – that of exploring how mathematical concepts live through various media – the chapter will touch upon the concept of function by articulating views of diagrams and instruments.

This will be pursued in accordance with a vision of mathematics as practice, for which focus on mathematical activity also comprises the design, production and use of tools/instruments the discipline has developed with. This is certainly true and commonly accepted for other disciplines like science or physics, where it is common practice that some experimental apparatus is assembled to verify specific conjectures or the observation through specific scientific instruments can be the source of new theories or phenomena.

Additionally, we will turn to Châtelet's (Châtelet, 1993/2000, 2010) inspiring work as he discusses many examples from the history of mathematics that are fundamentally entangled with diagrams: this will help us unfold the material and physical nature of mathematical concepts. In particular, we will dwell on those examples that we can directly relate to the concept of function.

The next paragraphs will disclose the reader that this choice is resonant with a consideration of concepts as devices, since (according to Châtelet) we can think of diagrams as kinetic capturing devices and of instruments as devices for making or working with/on mathematics.

Therefore, the rest of the chapter presents some examples of diagrams and instruments that support this vision. My intent is not to follow a logical-chronological account, rather to capture some elements that might be exploited as singularities in complex processes of concept formation. Far from considering these singularities to be exhaustive with respect to the current line of thought, the discussion will illuminate particular aspects concerning the concept of function. While highly situated, the examples attempt to attend at the complexity of the concept, finding a complementary path to that concerning concept definitions and discursive constructions. In particular, all the singularities we discuss are grounded into the wider idea of movement, which constitutes our lighthouse in navigating the concept of function.

1.3 Concepts and diagrams

The expert on ancient mathematics Reviel Netz (1998) argues that Greek mathematics is mainly concerned with diagrams: "that is, the Greek perception is that the object of mathematics is the diagram" (p. 38). Netz observes that in ancient Greece no mathematical notation had been developed: instead, the diagram was that inter-subjective object which made the building of the discipline possible. This is also reflected into how mathematical objects are defined in Greek mathematics: determination is reached through the diagram, as it is not only supplementary material for the constitution of an object, but a necessary and logical component of that object. Then, with modern mathematics, propositions and proofs, through an axiomatic account of the discipline and the predominance of written language, have taken the lead in the practice of mathematics, replacing the diagram in its role. This is probably the reason why diagrams in mathematics are considered to be subjugated to symbolic (letter) notation and language in general, as if they were infantile version of abstract mathematics. In this section, we draw again on the work of Châtelet (1993/2000, 2010), who elaborates on the role of diagrams in mathematics, shedding light on the gesture-diagram interplay and on the creative dimension of diagrams, in relation to the activity of mathematicians. According to Châtelet, the creation of new mathematics has its origin in the diagrams and genuine doodling of the mathematicians. His vision is particularly interesting as he illuminates the diagram as site of mathematical invention (or discovery).

The relegation of this part of the history of mathematics as mere "biographical traits" of mathematicians (like Galileo, Newton, or Leibniz) hides the invention under the carpet in favour of theorems and their display in textbooks (Knoespel's introduction in Châtelet, 1993/2000). Knoespel also cites the first edition of *Principia* (1687)[8], in which Newton claimed that geometry is "founded in mechanical practice" (p. xvii). Even though he had a true obsession with the ability to produce drawings and he considered such ability a necessary condition to comprehend the laws of nature, somehow this obsession reinforces the idea of a strict entanglement of drawing and mathematical inventiveness.

Knoespel also lists the major features and functions that Châtelet ascribes to diagrams:

- diagrams constitute technologies that mediate between other technologies of writing;
- diagrams create space for mathematical intuition;
- diagrams are not static but project virtuality onto the space which they seek to represent;
- diagrams represent a virtual strategy for entailment;
- diagrams are mediating vehicles which means they cannot only be recovered but rediscovered;
- diagrams have a pedagogical force that could be integrated into mathematical education.

Diagrams are indeed a crucial (not optional) aspect in the creation or invention of new mathematics if we think of mathematical concepts as physico-mathematical entities[9].

In light of this vision, we now dwell on some examples of diagrams analysed by Châtelet that point out essential aspects of function.

1.3.1 Oresme's diagrams

A type of diagram that gave fundamental contribute to the development of the concept of function was created by Nicolas Oresme (1320-1382), who was a French bishop during the Middle Age. His best-known work on the theory of forms' latitude is "*Tractatus de configurationibus qualitatum et motum*" (wrote around 1350). In this work, Oresme studied the motion of objects capturing velocity and time into *configurationes*, specific diagrams that combine latitude and longitude into a single graphical representation. In more general terms, within a *configuratio*, a vertical line represents the latitude of a quality, while a horizontal line represents its longitude: combined, they capture a change in intensity of a quality (e.g., velocity, temperature, etc.). As Clagett (1968) explains:

> "the base line of such figures is the subject when we are talking about linear qualities or the time when we are talking about velocities, and the perpendiculars raised on the base line represent the intensities of the quality from point to point in the subject or represent the velocity from instant to instant in the motion. The whole figure, consisting of all the perpendiculars,

represents the whole distribution of intensities in the quality, i.e., the quantity of the quality, or in case of motion the so-called total velocity, dimensionally equivalent to the total space traversed in the given time." (p. 15)

The variation of longitude accompanies the latitude and represents the division of a time or space interval (Boyer, 1968). Oresme faced the problem of *movement of movement*, calculating the distance travelled by an object when its speed is continuously changing.

In Oresme's diagrams, the intensity of a motion (a velocity) would be represented by its latitude, on the vertical line, and its time or duration would be represented by its longitude, on the horizontal line. This means that the area under a curve is a distance, and the different types of motion (uniformly difform, i.e. with constant acceleration and difformly difform, i.e. with variable acceleration) are *deformations* of a standard rectangle (which represents a uniform motion with constant speed, since every uniform quality has equal intensity in all of its parts; see Figure 1.1).

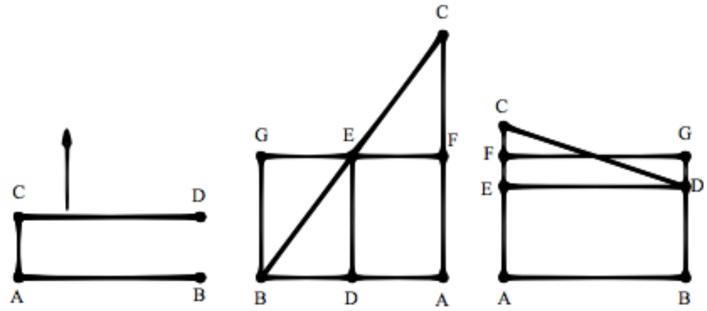

Figure 1.1. Oresme's diagrams: a segment (starting from C) representing the speed moves horizontally according to the considered time interval (AB). The movements are characterised by the deformation of the standard rectangle (left) (from left to right: constant speed, constant acceleration and constant deceleration) (Lagacé, 2015, my translation)

According to Châtelet, Oresme transformed the concepts of rectangles and triangles and at the same time these geometric concepts were changing the notions of time, distance, displacement, while also enriching them. The figures are not static but change in form according to changes in speed: they acquire a degree of mobility and plasticity, and actively partake in the comprehension of dynamic phenomena, instead of simply being the final product of the process of grasping a phenomenon (Sinclair, de Freitas, & Ferrara, 2013).

Oresme himself stressed that his *configurationes* allow to better recognize and to examine more quickly the uniformity (or difformity) which is somehow visible and apparent through the diagram, a sensible medium: "[This is true] because something is quickly and perfectly understood when it is explained by a visible example. [...] the imagining of figures is a great help in the understanding of things" (Oresme, quoted in Clagett, 1968, p. 175)

According to Boyer (1968), with Oresme's diagrams, concepts like speed came to be expressed quantitatively in terms of limits of ratios – that is, simply as numbers. Boyer also writes:

> "The terms "latitude" and "longitude" that Oresme used are in a general sense equivalent to our ordinate and abscissa, and his graphical representation is akin to our analytic geometry. His use of coordinates was not, of course, new, for Apollonius and others before him had used coordinate systems, but Oresme's graphical representation of a variable quantity was novel. He seems to have grasped the essential principle that a function of one unknown can be represented as a curve, but he was unable to make any effective use of this observation except in the case of the linear function. Moreover, Oresme was chiefly interested in the area under the curve; hence, it is not very likely that he saw that every plane curve can be represented, with respect to a coordinate system, as a function of one variable." (p. 291)

As Mader (2014) points out, Deleuze is critical to this observation: Oresme's longitude is not a coordinate. It does not coorder, but surprisingly, composes intensities and extensities into a surface area. In Oresme's configurationes, lines are not producing points or lines, but yield an entire area.

> "To consider the length as an area is to make clear that the cooperation of the two measurements involves the invention of a continuum capable of presenting as contemporaneous that which appears as already divided and that which asserts itself as an undivided entity." (Châtelet, 1993/2000, p. 44)

Coordination is a form of reduction, while composition is not. Châtelet also explains:

> "The length is not obtained only by putting standard measures end to end - that would be a simple accumulation - but mobilizes itself and makes it obvious that a dimension emerges, heterogeneous to the time parameter. Moving a line above a mobile subject to sweep over a surface invites another type of operation than that of the simple juxtaposition of bits of space that have already been cut out: it would be better to speak here of a *coalition* of stripes generating this or that surface and realizing a *device for summing degrees*. In combining verticality and horizontality, these devices are not content just to 'give resilience' to extended space, but enable us to watch the progressive acquisition of degrees of velocity and even immediately to

> obtain significant results concerning the 'uniformly deformed motions' without knowing any differential calculus" (p. 41, *emphasis in the original*).

Kaye (1998) shares a similar vision proposing that Oresme's diagram was a device to demonstrate geometrically that quantity (extension) and quality (intension) were bound together in a dynamic proportional relationship. Nevertheless, many scholars agree that, with his diagrams, Oresme anticipated Galileo by formulating the mean speed rule in a geometric way and he gave fundamental contribution to later representations of physical problems through graphical models.

In sum, Oresme's diagrams are *dynamic devices* that allow quantities to be given a double expression, in both extensive and intensive terms, without merging the two together. His figures are an excellent example of the creative power of diagrams and their openness to allude to new ideas and provoke new thought. They are not simply a way of organising space, but a way to create out the new from the composition of intensities and extensities. Châtelet's analysis of Oresme's diagrams allows to see that they are a powerful device, which was able to capture varying relationships by means of a visual solution. His analysis goes further in this direction, showing that "the coalition of several degrees implies a *simultaneous* grasp of these degrees, and this, within a *single subject*" (p. 42). Moreover, it shows that Oresme is combining two types of motion that, in the diagram, function as corresponding figuration of the progressive unfolding of the intensive and extensive:

> "How the acquisition of a quality is to be imagined. Succession in the acquisition of a quality can take place in two ways, according to extension and according to intension ... And so extensive acquisition of linear quality is to be imagined by the motion of a point flowing over that subject line, so that the part [of the line] traversed has the quality and the part not yet traversed has not the quality. Example: if point C were moved over line *AB,* whatever part was traversed by that point would be white and whatever was not yet traversed would not yet be white." (Oresme, quoted in Châtelet, 1993/2000, p. 43)

The cooperation among the two orthogonal movements (measurements), says Châtelet, "involves the invention of a *continuum* capable of presenting as contemporaneous that which appears as already divided and that which assert itself as an undivided entity." (p. 44, *my emphasis*). This aspect, according to Châtelet, also revealed the profound comprehension that Oresme had about the idea of *parameter*, as that which is unfolded not only horizontally (in the extension) but also as that dimension in which the 'spreading out' of consecutive degrees is realised.

> "Intensities allow no such thing as an indifferent juxtaposition: they do not add themselves together: they arrange one another, they increase or lessen one another. A degree that is inferior to another is not included in the latter as a part might be in a whole" (p. 40)

Oresme's diagrams therefore open a window on the virtual nature of mathematical concepts, while also shedding some light on the power of diagrams in the historical unfolding of fundamental aspects for the concept of function. His *configurationes* are fundamentally dynamic, in their being oriented to capture (not represent) the intensive nature of motion.

1.3.2 The infinitesimal triangle

Since the concept of function is considered to be the bedrock for the calculus, we now focus on another example that relies more closely on this mathematical field. It is well known that the history of Calculus includes developments from the early methods of the Greeks to compare areas and volumes, through the 'prime and ultimate ratios' of Newton and the infinitesimals of Leibniz, on to the formal epsilon-delta definitions and proofs of mathematical analysis (Tall, 2010). As already mentioned at the beginning of the chapter, the kind of solutions created by Newton and Leibniz to problems on curves used the blending of algebra and geometry, together with the introduction of variables and their relationships by means of equations (Kleiner, 1989). Even if the problems are solved through the creation of differential calculus, which is highly based on symbolic representations using letters, a fundamental role is still played by figures and curves. The contributions of Newton and Leibnitz are equally important is such discourse, but I have chosen to focus on the discussion on the infinitesimal triangle of Leibniz as it is put forward in Châtelet's work.

Châtelet (2010) describes the infinitesimal triangle in relation to the theory of virtuality, showing how it does not "really" exist insofar as we consider the whole family of virtual triangles that are infinitely close to it. Although mainly oversimplified in education, the idea of the infinitesimal triangle plays an important role in the theory/philosophy of Leibniz, for its nature is quite indeterminate: it is not a rigid figure placed in space, rather is a mobile entity changing its appearance together with the curve to which is related. It is all mobility. The concept of differential arises from this subtle and never-ending movement that characterise the function, the triangle and its shape.

Therefore, the infinitesimal triangle of Leibniz is: on the one side, an example of virtuality as it is a completely new mathematical idea, that escapes classical definitions and boundaries; on the other, crucial to the development of a new line of thought in mathematics, that is, the idea of differential. The type of movement tied to the idea of virtual triangle speaks directly to the concept of function as it focuses on *how the function changes*, which was the main aspect of interest in the early stages of calculus. Additionally, the triangle is considered not just in light of its fixed nature as a rigid figure, but essentially as a mobile way of treating the curve. We can imagine the triangle to be changing its sides as we move along the curve, together with the changing curve slope. At the same time, the infinitesimal triangle *is* the multiplicities of triangles that live close to him and that belong to the same family that characterises the curve. We can imagine those triangles to be arising from little modifications and displacements of the initial triangle (Figure 1.2), but, as Châtelet (2010) points out, this would reduce completely the idea of virtuality to an "accroissement petit" (little increase), that is, to the possible. What instead constitutes the nature of the infinitesimal triangle is its openness to be stretched, modified, moved.

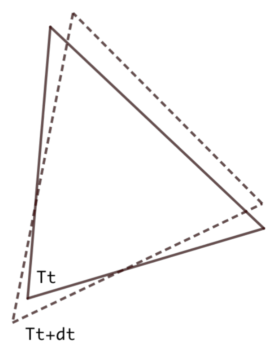

Figure 1.2. The infinitesimal triangle of Leibniz (redrawn as originally in Châtelet, 2010)

Summarising, we can see how the infinitesimal triangle is an example of materiality that do partake actively in the history of formation of the concept of function in calculus (and, later on, in the development of differential geometry).

The two examples that I have proposed shed some light on the concept of function through the discussion of specific diagrams that emerged in the course of history. The work of Châtelet helps provide insights into how ideas of movement and change, differently unfolded through dynamic figures, are fundamentally entangled with the concept of function and the way that mathematicians have tried to capture it.

Following a similar line of flight, I turn our attention to discuss the relationship between concepts and instruments.

1.4 Concepts and instruments

Mathematical tools[10] do not simply have a dialectical relationship with the discipline, being produced by humans to accomplish certain tasks in relation to a need or purpose (like in the case of the compass designed to draw circles). Rather, they essentially constitute the matrix from which they sprang, altering the practice, nature and conception we have of mathematics (Rotman, 2008). As an example, we might think of computer software that can calculate prime numbers with millions of digits[11].

This dialectical relationship might be scanned in the course of history for instruments as compasses, rulers, and early calculators, but a similar approach is also applicable to symbolic notations and graphical representations. One can easily find examples in the literature where such relationships are highlighted, for instance Bartolini Bussi and Mariotti (2008) discuss extensively the case of the compass.

In addition, the entanglement between mathematics and mathematical tools in the sense suggested by Rotman is even more apparent nowadays, as we see in everyday life a huge spread of computers, virtual and augmented reality devices, and digital technologies in general. We stress here that the ways in which mathematicians had interacted and interact with tools, and the ways in which the technologies shape how mathematics problems are solved and investigated, are meaningful in our discourse.

I want to offer examples, and – to this aim – I have chosen to focus the discussion on particular devices whose characteristics might be considered in-between those of a diagram and an instrument, namely nomographs and dynagraphs. This choice specifically draws attention to the ways of representing functions. I argue that unconventional modes to capture functional relationships might suggest and evoke peculiar aspects of the concept of function, which are meaningful and crucial for our discourse. The discussion of the two examples will provoke insights on the kind of perceptual and dynamic engagement that is involved in 'using' such instruments, with specific reference to meanings that emerge in movement with, on and around material arrangements.

1.4.1 Nomographs

A *nomograph* (or nomogram) is a device for calculating graphically mathematical relationships or laws (from Greek *nomos*, "law" and *grammē*, "line"). It is a bi-dimensional diagram that allows for the approximation of a function, given the variables that constitute the represented relationship. The foundation of this area of practical and theoretical mathematics is attributed to Maurice d'Ocagne (1862-1938) and many examples of nomographs can be found in astronomy books, where they were printed in large dimensions to foster precis calculations. Today nomography has been almost entirely forgotten but has been used extensively during the seventies for producing tools for engineers to compute fast, even though often limited to practical contexts or situations. Nomography still survives in the form of simple nomograms that are applied in medical contexts and they can be found in some science and engineering articles. The theory of nomography interestingly draws on many aspects of geometry and algebra as well as on other branches of mathematics, which are fused together in innovative ways and used in synergy for the design and creation of diagrams. Nomography developed contemporary to slide rules, which allow to perform basic arithmetic calculations and a wide set of equations following a sequence of steps. A nomograph instead is designed to solve a specific equation in one step. It can be observed that two nomograms for the same equation may appear very different from each other, since the final look mostly depends on the inspiration of the designers and their creativity (Doerfler, 2009). Moreover, there are example of nomograms that can be considered real pieces of art (e.g., see Figure 1.3): their beauty relies not only upon the creativity of the artist who brings them to life, but also upon the ease with which they make very complicated formulas solvable, by hiding in the design process all the most challenging "steps" to get to the solution.

The simplest example of nomograph is the kind of double-reading scale that can be seen on most thermometers, which usually contains reference to both Celsius and Fahrenheit degrees, one on each side of the mercury column. The two scales are juxtaposed in a way that allows for reading corresponding values of temperature degrees in one unit or the other. This is obtained starting from two equal number lines, then shifting and shrinking one of them according to the parameters that describe the relationships among the two temperature scales.

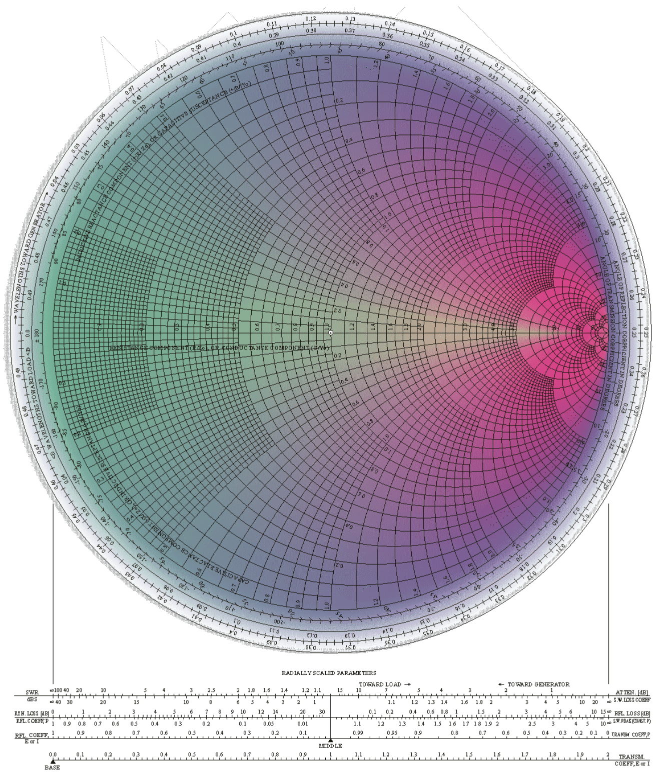

Figure 1.3. Smith Chart, a beautiful example of nomograph (retrieved from https://upload.wikimedia.org/wikipedia/commons/7/76/Visual_Smith_Chart.png)

It is often the case that a nomograph is built up with three scales, that is, with three variables that are somehow interrelated. For instance, Doerfler (2009) discusses the case of a parallel-scale nomograph that allow to calculate a value $f_3(w)$ as the sum of two functions $f_1(u)$ and $f_2(v)$, in which each function is plotted on a vertical scale using a suitable scaling factor (m_1, m_2, m_3). Then, a diagonal line called 'index line' or 'isopleth', is connecting the values to be summed up and intersecting the middle scale at the point which gives the solution. By comparing similar triangles in the configuration (see Figure 1.4), the scaling factors and the distances between scales (a, b) might be calculated. It is worthwhile mentioning that with little algebra manipulations and simple geometric relationships one can then create variations from this simple diagram. For example, negating a variable simply means to reverse the scale up or down; or using logarithmic rather than linear scales also allows to represent multiplicative or exponential equations.

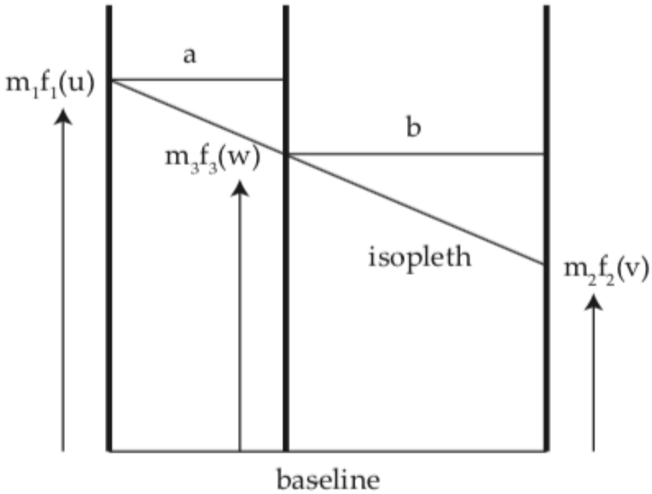

Figure 1.4. Derivation for a parallel-scale nomograph (Doerfler, 2009)

A nomograph can be thought of as in between a diagram and a tool, since it captures with its form and design a specific relationship among variables as much as a diagram would, while it can also be used as a tool for systematic calculation in a system of parallel coordinates. Moreover, even if it seems static as a diagram, the solutions are to be found in a dynamic way, that is, connecting the values for each variable that is to be composed.

Relationships between variables as expressed in the formula of a function are "visible" and "put at work" in the nomograph through the diverse scale of each line, representing one variable, and the distance among lines. More elaborate nomographs can require more than one isopleth to obtain the final value, but basically it is the design of the tool that embeds – through innovative shapes and appearance – the way that the variables are supposed to be in relation with each other, that is, their correspondence in terms of the function expressed by the nomograph.

To sum up, back to our discourse on function, nomographs were born at the conjunction of different mathematical fields, from algebra to geometry, but also arithmetic and calculus, and created a new way of capturing functional relationships that rely on practical situations or problems (through approximation). They work mainly with a system of parallel coordinates, which is the primary characteristics that consists of a deviation from the standard Cartesian representation of function. Using a nomograph to solve an equation suggests an input-output, algorithmic and applicative vision of functions. The emphasis is on how different values of two or more variables are interconnected through one or more segments. Despite the stativity of scales and numbers, there is underlying movement involved in the design (stretching, curving and manipulating scales) and finding solutions with the nomograph as well entails a particular way of coordinating the available scales and tracing segments.

1.4.2 Dynagraphs

The second example we investigate is that of dynagraphs. Like nomographs, dynagraphs are dynamic representations that work with a system of parallel coordinates. Dynagraphs are well known in mathematics education research field, since they were proposed by Paul Goldenberg, Philiph Lewis and James O'Keefe to introduce to students the ideas of variable and of functional dependency (Goldenberg, Lewis, & O'Keefe, 1992). Dynagraphs were pedagogically thought of as a bridge between input-output machine (see §1.5.1 for a discussion) and function graphs in the Cartesian plane. Indeed, a dynagraph consists of two parallel (horizontal) number lines, the x- and the y-axis, and it is configured so that dragging an input (a value along the x-axis) causes an output to move along the y-axis, according to a specific relationship (function) that connects the two variables. Whether this relationship is the linear function $y=3x$, for example, as the input is dragged toward the positive direction (right) the input moves to the right as well, at a speed, which is three times the speed at which the input is moved. In a symmetric fashion, the same happens with a motion that occurs to the left. Instead, when the dynagraph is not put into motion, one can only see the one-to-one relationship between the current values at which the cursors are positioned on the two axes (see Figure 1.5, which captures three instances of a dynagraph for a linear function). Therefore, in this dynamic representation: (1) the speed at which the point varies on the output axis (in function of the variation that occurs on the input axis) and (2) the relative positions of dependent and independent variables are the perceptual elements that suggests the given relationship, by means of 'how' the function behaves (how corresponding points are moving) in such coordinate system. This means that, whether we perform uniform motion of the input variable on the x-axis (dragging the cursor at a constant speed), we might guess the type of function that is at play in a specific dynagraph by looking at how the output cursor (and the segment that connects the two) moves.

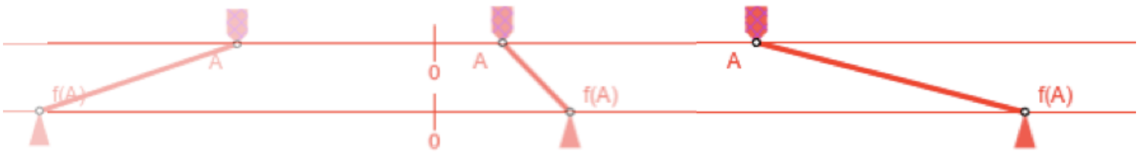

Figure 1.5. A dynagraph for the function y=2x. The picture shows three different positions, captured in three different time instants and characterised by different shading

I believe that using this device might also be intriguing and challenging for students that have already encountered functions and graphs. In fact, this way of capturing functional relationships challenges well-established graphical assets, through disrupting the usual manner of depicting functions in graphical terms and embedding the way in which a function changes through the coordination of two different kinds of motion.

Healy and Sinclair (2007) have analysed narratives about dynagraphs that reveal puzzlement in teachers when exposed to a dynagraph representing a step function: the jerky movement that results in the composition of the fluid movement on the *x*-axis and the movement "in fits and starts" on the *y*-axis evokes a particular style or quality that is associated to that function. In the paper, Healy and Sinclair report about this dramatic quality being recalled as the walk of a "cool dude" grooving across the screen or a function "with personality" and as particularly emerging from unexpected responses in terms of movement to the fluid dragging of the mouse. What matters here is that there is no analogous qualitative felt nuance in the canonical representation of the step function on the Cartesian plane, which mainly elicits static metaphors such as that of a 'stair'.

Following up on our discourse, we might draw on this to say that the dynagraph of a step function creates a *felt quality* about that function, which is not emerging in the same way through other representations.

Of course, in a dynagraph, also the way in which one can 'read' analytical features of a function (like domain, codomain, asymptotes, etc.) changes with respect to usual way exploited with graphs embedded in the Cartesian plane. As an example, in the case of standard Cartesian coordinates we would expect to recognize points of discontinuity (e.g. vertical asymptotes) as those points on the *x*-axis which has no corresponding point on the graph. Instead, in the case of the dynagraph reaching a point of discontinuity makes the cursor on the *y*-axis to 'run away' from the screen and disappear in one direction.

Kaenders (2014) discussed another version of dynagraph which is not dynamic in the way in which the ones presented until now are. Equally-spaced positions of the independent variable are chosen and connected to the corresponding value on the y-axis through arrows, as if specific positions of a dynamic dynagraph were crystallized in different frames and juxtaposed: Figure 1.6a shows one of these diagrams (here, the x-axis and y-axis are swapped with respect to the previous dynagraph, being at the bottom and at the top, respectively).

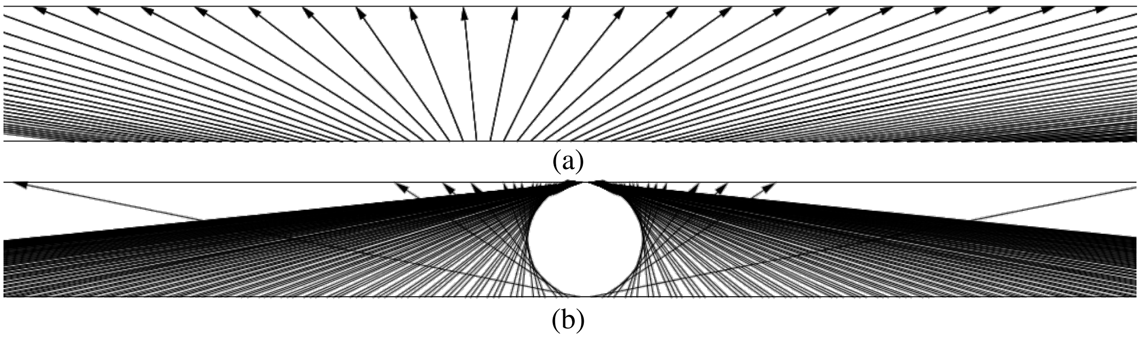

Figure 1.6. Different version of dynagraphs, (a) y= 3x; (b) y=a²/x (Kaenders, 2014)

Particularly interesting for our discourse is how the felt quality of the dynamic dynagraph still survives and animates the diagram through the shading created by contiguous arrows getting closer to each other, when the eye is moving 'towards infinity'.

To recall the vertical asymptote example presented in the case of dynamic dynagraphs, we might ask how we can spot an asymptote here. In the case of Figure 1.6b, the image shows the configuration for the function $y=a^2/x$, which presents a vertical asymptote for x=0 and a horizontal asymptote, y=0. The circle created as envelope of the arrows in a neighbourhood of zero is evoking two counterpoised forces balancing one another. That of the arrows closer to zero, quickly slanting towards (positive or negative) infinite direction; and that of arrows springing from positive and negative bigger values, whose extremities are densely piling up at the centre of the upper axis. No movement is really occurring there, but the diagram still alludes to it. Vertical translations of this function would result in diagrams that still maintain a 'hole', but the circle would be distorted, that is resulting more elliptical through the envelop of arrows, with tangent point on the y-axis corresponding to the value of the new horizonal asymptote.

Kaenders (2014) also discusses the way in which the composition of functions might look like with such dynagraphs. It is interesting to notice how this composition corresponds to the physical action of following the path created by consecutive arrows, provided that corresponding dynagraphs for the functions to be composed are parallel to each other and orderly aligned with respect to each other.

This example is far from giving an historical account for the concept of function, since dynagraphs have been recently developed and since they are mainly known within the field of mathematics education. Nevertheless, it is relevant for our discussion for at least two reasons. The first reason is that dynagraphs are fundamentally grounded in motion. Both static and dynamic dynagraphs are telling stories about functions through their capacity of suggesting variation, covariation of quantities, change and relationships among variables by means of real or evoked motion. Dynamic dynagraphs can be constructed and used are displayed thanks to digital technologies like DGEs, which are creating space for interactive manipulations, like dragging, therefore they constitute instruments that create new meanings and nuances for the concept of function.

The second reason concerns the fact that the dynagraph is an excellent example to show the reader how the ways in which a function is presented and explored are evoking peculiar aspects for that function, which other points of views, or material arrangements, may not grant. Through the examples, I tried to underline that dynagraphs are entailing different ways of seeing and displaying functions, as well as different actions and qualities from those elicited by graphs embedded in the Cartesian plane.

These examples are highly situated and non-exhaustive of what the concept of function might be and become but help us mediate the issue of how diagrams and tools that we might relate to concepts are not subjugated to the ways in which we talk about them. Instead, they create new nuances in the ways that we can think of and envision functions. In addition, we might also argue that the diagrams and instruments we presented are not mere representations for (or to talk of) functions but are materially implicated in the way in which we can encounter the concept of function.

1.5 The concept of function in mathematics education

To close the chapter, we will now refer to the big corpus of literature in mathematics education that has been developed on the concept of function, about which there is more to read than one can possibly do in three years of a Ph.D. Nevertheless, we will frame the present work in a sufficiently wide panorama to position it into the current research studies. In particular, I chose to focus on the diverse studies that have tackled and described "functional thinking" as the way of making sense of the concept of function in its multiple dimensions, some of which have already emerged in the previous sections with the proposed examples. Finally, there will be a section devoted to deepening the line of research which is at the heart of this dissertation, namely the studies that centred on the use of specific technology to treat the concept of function via a graphical approach. Particular attention will be drawn to those studies that have considered movement (bodily movement, but not limited to it) as a key element in the constitution of mathematical ideas related to the concept of function.

1.5.1 A brief overview

My interest here is to outline some of the studies in mathematics education that have dealt with functional thinking, without taking into account the age of the students, which usually involve undergraduate or high-school students. My aim is to outline the main line of research as they are developed, highlighting the emerging themes.

One recurrent point is the relevance of the concept of function from both an epistemological/historical point of view, and a didactical point of view. Its presumed importance in the curriculum seems to arise from the foundational character of the concept for modern mathematics, as we tried to outline at the beginning of this chapter, and from the essential use of function in related areas of the sciences to model processes and phenomena.

Researchers in mathematics education have studied the teaching and learning of functions tackling the issue from different points of view. Thompson and Carlson (2017), for example, underline the importance of *covariation* as crucial for Newton's mathematics and consequently for the emergence of calculus as a body of thought (Kaput, 1994).

They distinguished the idea of covariation from that of correspondence, which is the prominent aspect in Dirichlet's definition. Thompson and Carlson (2017) argue that ideas of continuous variation and continuous covariation are "epistemologically necessary for students and teachers to develop useful and robust conceptions of functions" (p. 423). They additionally suggest that variational and covariational reasoning are fundamental to students' mathematical development, especially in relation to concepts like rate of change and slope.

In Thompson's (1994) theory of quantitative reasoning, a person reasons covariationally when she envisions two quantities' values varying and this is done while thinking of them varying simultaneously (as discussed in Thompson & Carlson, 2017). For example, thinking of a runner, who is running in space and getting farther from a reference point, to reason covariationally means to be aware that time is passing and that the runner is at some distance from her start at every moment of the elapsed time. Holding in mind the image of two quantities that vary simultaneously entails coupling the two quantities so that "in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299). Saldanha and Thompson (1998) add that covariational reasoning is developmental: the early stages involve coordinating two quantities' values, while "[a]n operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image" (*ibid.*).

Carlson and colleagues (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) defined "*covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354).

Other researchers have also underlined that analysing, manipulating and understanding relations among varying quantities are crucial to understand functions (Jere Confrey & Smith, 1994, 1995). Nevertheless, the more conventional approach to functions seems to be a *static* one, that is so say that conventional treatments of functions typically start by creating a rule of correspondence between x-values and y-values, by establishing an equation of the form $y=f(x)$ (Confrey & Smith, 1994). Such perspective is based on a relational view of function, namely on the causal relationships between input-out pairs of values.

Vinner and Dreyfus (1989) used and refined the framework of Tall and Vinner (1981) to analyse 10^{th} and 11^{th} grade students' questionnaires looking for the main *concept definitions* of function that emerged. They divided these definitions into six categories, according to the fact that function was described by students as a rule of correspondence, a dependence relation, a rule or relation more generally, a formula or equation, or a (graphical or symbolical) representation. In the last category, in particular, the function is often identified with a graph, with the symbolic expression $y=f(x)$ and with "the two potatoes with arrows" diagram.

Thompson (1994) built as well on the notion of image, but with a slightly different focus, namely on image as dynamic, originating in bodily actions and movements of attention, and as the source and carrier of mental operations, to analyse the students' understanding of integral and derivative.

> "The development of mature images of rate involves a schematic coordination of relationships among accumulations of two quantities and accruals by which the accumulations are constructed. For example, in the case of constant speed, the total distance traveled in relation to the duration of the trip can be imagined as each having accumulated through accruals of distance and accruals of time so that at any moment during the trip the total distance traveled at that moment in relation to the total time of the trip is the same as the accrual of distance in relation to the accrual of time" (p. 5).

Thompson (1994) also suggests that rates involving time are the most intuitive, as variation is conceptualised as happening "across-time", while a further level of abstraction is needed to develop an image of rate with two different (non-temporal) variables.

David Tall and colleagues discussed extensively the idea of cognitive roots in relation to the concept of function (Tall, 2010; Tall, McGowen, & DeMarois, 2000). A cognitive root is defined as an anchoring concept that entails the creation of a solid background for a student learning mathematics that allows for the development of higher levels of abstraction. An example of a cognitive root is the notion of "local straightness" in calculus, as Tall (2010) shows that it contains the potential seeds for understanding the concept of differentiation. Concerning the concept of function, Tall and colleagues (2000) have also investigated the function machine as an input-output box as candidate for a cognitive root for the concept of function. It "embodies both its process-object duality and also its multiple representations" (p. 4), namely it can be properly related to a "procept" (Grey & Tall, 1994). The function machine embodies both an object and the input-out process that pertains to it, leaving in the shadows the relationship that is established between input and output.

This idea was first investigated by Slavit (1997), who distinguished between an operative vision of function and a property-oriented view on function based on student's conception of function relative to functional growth and covariance. His framework wanted to extend previous work on functional understanding, e.g. the covariance view, by posing less emphasis on the manner in which variables are changing and more on the properties that emerge through observing those changes. In the case of function, saying that one has a property-oriented view of the concept would entail that she can describe all the features associated to an object that satisfy the definition of function. This in turn, according to Slavit, means that she can "understand the concept of function by transforming [her] experientially-based perceptual patterns of functional growth behaviors into well-formed understandings of specific functional attributes" (p. 261).

This brief literature review about functional understanding has outlined initial studies on the concept of function that are considered pivotal in the field. These studies mainly seem to deal with a contrast between a dynamic/processual and static/structural conception of function, which has been characterised by Davis (1975) as the "process-product dilemma", namely the students' difficulties in managing a function as both an algorithm, or a process, and a number, or the result of that process. The contrast seems to significantly stem from the notion of variable, whose generalisation involves considering several processes simultaneously. For example, the distinction between covariation and correspondence, which has been already introduced, is also related to such static/dynamic opposition. Monk (1992) enters the discussion by distinguishing two different conceptions of functions: "Across-Time" and "Pointwise". Moschkovich, Schoenfeld and Arcavi (1993) put forward an aspect of Pointwise conception, introducing the notion of "Cartesian Connection" (the ability to envision solutions to a linear equation as ordered pairs or points in a Cartesian graph).

Sfard (1991) proposed the distinction among structural and operational conceptions of mathematical concepts as complementary in mathematics. The dual nature of mathematical constructs arises from verbal descriptions and chosen representations for envisioning that concept. In the case of the concept of function, the graph encourages a structural approach, while the algebraic expression, according to Sfard, can be easily interpreted both operationally, "as a concise description of some computation, or structurally, as a static relation between two magnitudes" (p. 6). The dual nature – process versus object – of mathematical concepts, which insists on the opposition dynamic versus static, is approached by Sfard considering the latter as a static/structural reification of the former.

Some of the framing concepts discussed in the studies above, like those of image, abstraction, and multiplicative object, are rooted in a Piagetian perspective, which implicates subsequent steps or levels have to be achieved to acquire knowledge about function. Notwithstanding the studies have highlighted important aspects related to how students might conceive the concept of function, these are not so much in line with our vision of concepts. The discussion will now shift to another line of research, namely the studies that have specifically focussed on representational and graphical aspects of functions.

1.5.2 Graph, graphing and graphical approaches

This subsection wants to bring forth ideas from some of the studies in the field that have considered the learning of functions by putting attention to graphical approaches, especially in the context of technological environments and/or considering the role played by the human body in graphing activities.

Leinhart and colleagues (1990) noted that "although functional relationships have been recognized for some time as important constructs in the development of abstract mathematical knowledge (Piaget, Blaise-Grize, Szeminska, & Bang, 1977), functions and graphs have not been the object of much intellectual scrutiny by the educational community" until the '90s. Their review proposes an extensive report on those research studies focussed on the interpretation and construction tasks associated with functions and graphs (or other kinds of representation). They also started a discussion on the role of technology, especially with respect to the affordances of emerging technology to connect multiple representations. Only a decade later, as the technological developments were rapidly creating new possibilities for mathematics educators and new learning environments, Kaput and Roschelle (1999) wrote:

> "we see new technologies creating a possibility to reconnect mathematical representations and concepts to directly perceived phenomena, as well as to strengthen students' understanding of connections among different forms of mathematical representation. By starting from more familiar antecedents, such as graphs and motion, both in kinesthetic and cybernetic form, and developing towards more compact and formal mathematical representations, we see an opportunity to create a new path of access to mathematics that has too often remained the province of a narrow elite." (p. 23)

In fact, Kaput created the microworld *SimCalc*, a software application that allows students to work with relationships about time, velocity and positions through manipulating different representations (simulation, equation, graph, …) as a way of democratizing calculus, by making the mathematics of change accessible to lower secondary school students (Kaput, 2000). Many studies have been developed since then, most of which confirm the support that this kind of linked representations give to students in grasping relationships among functions and movement throughout the happening of some phenomena (e.g., Hegedus & Moreno-Armella, 2009).

Following a similar line of research, Schwarz and Dreyfus have investigated students' acquisition of the function concept in the setting of their *Triple Representation Model* (Schwarz et al., 1990; Schwarz & Dreyfus, 1995) software, which allows for dynamic interplay between graphic, numeric, and equation settings. Their research suggests that the actions performed both on the function itself and on its representations allow the student to progressively investigate and pinpoint characteristic properties of that function.

A group of Israeli colleagues (Schwarz & Yerushalmy, 1992; Yerushalmy & Schwarz, 1993) has worked extensively on a function-based approach to algebra aimed at connecting different representations for functions (tables of values, graphs, symbols, words) through the use of technology. In particular, Yerushalmy (2001) focuses on a curricular sequence of three phases: "(1) emergence of the concept of function throughout modeling, (2) manipulating function expressions and function comparisons (equations and inequalities), and (3) exploring families of functions and specifically linear and quadratic functions." (p. 126) As a development of the same project, Yerushalmy and colleagues have also created an environment that enables students to explore how formal representations change in relation to the hand movement (moving the mouse on the computer screen) and have further investigated, among other aspects, the role of bodily activities in mathematical modelling motion, through the *source-path-schema* by Lakoff & Nunez (2000) (Botzer & Yerushalmy, 2006).

One line of research specifically centres on students' understandings of functions and graphs as expressions of embodied cognition (Monk & Nemirovsky, 1994; Nemirovsky, 1994; Nemirovsky, Kelton, & Rhodehamel, 2013; Nemirovsky & Monk, 2000; Nemirovsky, Tierney, & Wright, 1998). These studies have analysed students' experiences in the context of graphing motion, through detailed phenomenological oriented commentaries, to elucidate how one comes to inhabit a mathematical environment, or to put it differently, a *lived-in space* for herself.

Nemirovsky and colleagues (1998) have described two girls graphing motion through the use of a motion detector, characterising their lived-in space as “relational, intentional, and creative” (p.153). Their approach has been enlarging the perspective on functional thinking, incorporating the ways in which the students become familiar with graphing activities and the resources they can gain from resemblance with known situations and experiences, as contributing to their mathematical understanding.

Nemirovsky and Monk (2000) have introduced the idea of “fusion,” to speak of the indissolubility of form and meaning in relation to symbols (e.g., a graph). In relation to graphs, they have shown how the fusion experiences pertained to both the researcher and the interviewed girl, so fusion it is not replaceable with the inability of separate a symbol and its referent but is “an engagement into the world of make-believe that animates symbols and makes them meaningful” (p. 196).

Nemirovsky and Tierney (2001) described activities for 4th graders on the mathematics of change, focussed on generation and interpretation of graphs, number tables and stories of events the students are familiar with. They propose that learning graphing “entails developing the capacity to ‘direct seeing’ (i.e. without intermediate inferences and calculations) events and qualities dwelling in symbolic expressions; a development that involves intricate experiences of seeing-as, recognizing-in, interpreting emptiness, and animating homogeneous spaces.” (p. 99)

More recent research has been developed to account for the role of graphing motion technology in such activities, proposing that “the development of tool fluency entails the interpenetration of the perceptual and motor aspects of an activity, allowing the performer to act with a holistic sense of unity and flow” (Nemirovsky et. al., 2013, p. 373). The achievement of such interpenetration is referred to as *perceptuo-motor integration*, and the authors proposed that “perceptuo-motor integration can be the basis of a type of mathematical learning in which we generalize by means of enacting bodily orientations and give meaning to symbols by awakening perceptual and motoric patterns.” (p. 406). The presentation of two detailed case studies in the context of a mathematics exhibition shows the gradual transformation occurring in the participants’ movements with the tool and fostering their understanding about parametrical functions, which are at play in the context of graphing motion activity with the Drawing in Motion machine[12]. This study shed considerably light on how the "felt or lived qualities" made available to participants a nuanced situational sensitivity to relative positions, velocities, and so on. Specific attention has been drawn to the role of gesture, multimodal engagement and imagination in these and similar activities (Ferrara, 2014; Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013).

We opened the chapter by investigating what the concept of function might be. After the review offered in this last section, I want to go back again to the work of Thompson & Carlson (2017). They write:

> "It is important to note that we said that a function is a conception. A function resides in someone's thinking, so the nature of a conceived function is relative to the person conceiving it. Notice also that we did not specify a particular way in which a person conceives that quantities' values vary, nor did we specify the way in which a person conceptualizes covariation. The researcher or teacher who claims that someone has conceived a relationship between quantities as a function must describe the way this person has conceived that values vary and the way in which they covary, otherwise the claim is vague. Also, a researcher must describe the person's conceived domain and range (the values that the person envisions quantities or variables having), which will depend on the individual conceiving the function." (p. 444)

We have touched on many different perspectives and approaches in mathematics education that have offered complementary visions to further discuss how the learning of function might occur. We have specifically focussed on some of the studies that have investigated the role of representations and graphs in such process, dwelling specifically on graphing motion activities.

My interest here is to follow this last line of research, as it is offers new lines of thought to describe and better understand how body, tools and graphs create meanings for the concept of function, disentangling the concept from the monolithic vision of it as a personal conception. This aligns with the attempts of section previous section (§1.3 and 1.4), which aim at illustrating examples of diagrams and instruments that have mobilized and created new nuances for the concept of function. We also began investigating the role of movement both in terms of bodily orientations and from the point of view of onto-epistemological concerns, which will be further elaborated in the following.

[1] A *dynagraph* is a kind of dynamical graph that uses parallel axes to display a function by means of a dynamical representation. Dynagraphs will be discussed in more depth in §1.4.2, so that the reader who is not familiar with dynagraphs will have the opportunity to delve into them in the following.

[2] The process of *saming* is defined by Sfard as "the act of calling different things with the same name. […] Saming is thus the act of associating one signifier with many realizations." (Sfard, 2008, p. 170). These realizations thus can be then approached with narratives on their common signifier, narratives that are isomorphic to each other. According to Sfard, the fact that isomorphisms between the three different contexts were identified was crucial to the emergence of the function concept.

[3] "Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation* in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x. We give the name of *function* to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the *value* of the function at the element, and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the *same* function." (Bourbaki, 1939, quoted in Bottazzini, 1986, p. 7). Bourbaki also furnishes the definition of function as a subset of the Cartesian product $E \times F$, namely as set of ordered pairs.

[4] In 1834 Lobachevsky writes: "The general concept of a function requires that a function of x be defined as a number given for each x and varying gradually with x. The value of the function can be given either by an analytic expression, or by a condition that provides a means of examining all numbers and choosing one of them; or finally the dependence may exist but remain unknown." (quoted in Medvedev 1991, p. 58). In 1837 Dirichlet writes "If now a unique finite y corresponding to each x, and moreover in such a way that when x ranges continuously over the interval from a to b, $y=f(x)$ also varies continuously, then y is called a continuous function of x for this interval. It is not at all necessary here that y be given in terms of x by one and the same law throughout the entire interval, and it is not necessary that it be regarded as a dependence expressed using mathematical operations." (quoted in Medvedev 1991, pp. 60–61).

[5] Many of the terms in this last sentence need explanations. Oversimplifying, virtual concerns what is latent or potential in an entity and is opposed to the possible as the latter is already predetermined, while the former is genuinely indeterminate. The concept of virtuality will be unfolded in the next chapter (see Chapter 2), specifying also how it might be understood in relation to the concept of multiplicity rather than in essentialist terms. Concerning the idea of concepts as material arrangements as well as the idea of mathematics as practice in the sense of Rotman (2006), see the discussion later in this chapter (§1.2).

[6] See Chapter 2 for an in-depth discussion of the virtuality of mathematical concepts.

[7] This example draws on de Freitas and Sinclair (2014), who similarly discuss the first vision of the circle as 'realizing the possible', while the second as 'actualizing of the virtual'. Chorney (2016) discusses the circle in a similar fashion, proposing that "as with any mathematical object, [a circle] ought not be seen as a reproduction of an ideal form, but rather as a meshwork of materials and forces" (p. 45).

[8] Isaac Newton, "Preface to the First Edition of the *Principia*" in *Newton's Philosophy of Nature: Selections from his Writings* (New York: Hafner Press, 1974 [1953], 9-11; 10), as cited in Châtelet (1993/2000).

[9] As we have already mentioned, Châtelet (2010) proposes a fundamental co-provocation between mathematics and physics, naming it *physico-mathematique*, which is about "finding a correspondence (balance) between the virtualities of a thing and the experimental systems with which I can make [that object] explode" (p. 11, my translation).

[10] In this section, and in the entire chapter, I am using the words 'tool', 'instrument' and 'device' more or less as synonymous. I am not drawing on any particular framework to ground my considerations and when this is done, the reader can find the corresponding references in the text.

[11] GIMPS, for example, is a collective of volunteers who use a freely available software to search for Mersenne prime numbers since 1996. $M_{77232917}$ is the last Mersenne number found with GIMPS.

[12] The Drawing in Motion machine was developed and manufactured by the Oregon Museum of Science and Industry and it is a mathematical instrument that requires the collaboration of two users. Each user "controls the motion of a handle along a 3-foot linear scale, corresponding to a graphical vertical or horizontal axis. […] A large LCD screen displays a cursor controlled by the two handles that determine the *x*- and *y*-coordinates of the cursor." (Nemirovsky et al., 2013, p. 382). By moving one handle each, the two users can draw a line on the screen. This device is the digitalized version of the Drawing Machine, which allows for a the creation of a two-dimensional curve from the composition of two orthogonal movements through a (Noble, DiMattia, Nemirovsky, & Barros, 2006).

Katy Ann Gilmore
Fold Number 1, 2017
Acrylic on dibond
44 × 46 in
111.8 × 116.8 cm

2

The Mathematics of the Virtual

This chapter is devoted to unfolding the philosophical concept of the virtual and its relationship with mathematics and mathematical concepts.

In Chapter 1, I have already touched on the concept of virtual in relation to mathematical concepts. However, some issues have only been evoked, to begin troubling the reader with theoretical arguments regarding the ontology of concepts. Now it is time to enter deeper into the argument, as it will be the ground soil to advocate for the mobility of mathematics. The present chapter mainly draws on: (1) the work of the philosopher and mathematician Gilles Châtelet (1993/2000, 2010) and (2) on Manuel DeLanda's reading of the realist ontology of Gilles Deleuze (DeLanda, 2002). This is the chronological order in which these resources appeared, and DeLanda's argument is also sustained by his reading of Châtelet's work. Nevertheless, the chapter will first introduce the metaphor of the virtual as developed by DeLanda in his book "Intensive Science and Virtual Philosophy", whose content has inspired the title of the present chapter. Deleuze and Châtelet knew each other and the first influenced the work of the second in a significant way. They share a similar philosophical substratum, so we will first introduce the virtual from the standpoint of the philosophical perspective of Deleuze. Then, we will dwell in the specificity of the virtuality of mathematical concepts.

2.1 Multiplicities versus essences: Manifold as a metaphor for the virtual

In the attempt of characterising Deleuzian ontology, DeLanda turns to the mathematical concept of manifold, reconstructing the example proposed by Deleuze (1990, 1994).

Deleuze pursues a realist ontology, that is, one that grants "reality full autonomy from the human mind, disregarding the difference between the observable and the unobservable" (DeLanda, 2002, p. 2). DeLanda aims at describing Deleuzian ontology underlying the difference between essences and multiplicities. Essences constitute the world of the Ideas in the Platonic sense, while the concept of multiplicity grounds the ontology of the Deleuzian realism.

DeLanda claims that the "essence" of something is that which explains its identity, its fundamental characteristics. Essences provide things to be unified and to possess a timeless identity. On the contrary, a multiplicity lacks unity and implies a progressively defined identity, which is not given all at once. Whether we speak of essences, we are forced to bear to their instantiations the same relations which a model has to its copies. Consequently, a relation of greater or lesser resemblance with essences allows explaining similarities and resemblances among things. This is no longer true in a world populated by multiplicities, as we should explain in the following.

The distinction between essence and multiplicity as immaterial entities that are responsible for the genesis of form, is necessary to grasp the argument within the philosophy of Deleuze. Essences imply that *matter is a passive receptacle for external forms*. On the contrary, "*multiplicities are immanent to material processes spontaneous capacity of generating pattern without external intervention*" (DeLanda, 2002, p. 28, *emphasis in the original*). This implies that, in a Deleuzian ontology, natural kind "is not defined by its essential traits but rather by the *morphogenetic process* that give rise to it" (pp. 9–10, *emphasis in the original*). Multiplicities play a pivotal role in the mutual arrangement and articulation of the three ontological dimensions in a Deleuzian world: the virtual, the intensive and the actual. To understand this, we shift attention to the metaphor of manifold as DeLanda discusses it.

2.1.1 Multiplicity and manifold

Following the history of mathematics, DeLanda shows the historical origins of the concept of manifold, which is crucial to an explanation of the concept of multiplicity. DeLanda goes back to the development of analytical geometry by René Descartes and Pierre de Fermat, which allowed mathematicians to face a variety of new physical problems based on curves and trajectories that the geometrical methods from the Greek traditions were not able to solve. Once embedded in a Cartesian plane, with fixed axes, every point on a curve could be expressed as relations between numbers. In particular, in this first shift, the new algebra resources could assist the solution of geometrical problems.

The second relevant shift is that which brought to life the differential geometry of Friedrich Gauss and Bernhard Riemann. According to DeLanda, the "basic idea was the same: tapping into a new reservoir of problem-solving resources, the reservoir in this case being the differential and integral calculus." (DeLanda, 2002, p. 11). This meant that curved lines or surfaces, thanks to the concept of *rate of change*, could also be characterised by the rate at which some property changed in-between different points (e.g. curvature). Gauss then used the fact that calculus allowed to work with local information about the surface to develop a method for "coordinatizing" the surface, that is, using a reference system implanted on the surface itself. This implied that the surface could be studied "*without any reference to a global embedding space*" (p. 12, *emphasis in the original*).

Briefly speaking, the modern conception of manifold defines a n-dimensional manifold as a smooth object that *locally* looks like the Euclidean space $\mathbb{R}^n$. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n. For example, in a circle, which is a 1-manifold, every point lies on a small curve that looks like a line segment ($\mathbb{R}^1$). Similarly, a torus and a sphere are 2-manifolds, and every point lies on a small slightly curved region which looks like a plane ($\mathbb{R}^2$): if we cut out a small piece of either surface and "zoom in", it would look like a limited portion of a plane. Manifolds can therefore be equipped with additional structure. As anticipated, since the manifold can be considered to be locally equivalent to some Euclidean space, differential relations could be used to characterise relations among points on the surface[2], and it is possible to define paths (trajectories for points) on the manifold. As a starting point for unfolding the metaphor of multiplicity as manifold, DeLanda observes:

"As the mathematician and historian Morris Kline observes, by getting rid of the global embedding space and dealing with the surface through its own local properties 'Gauss advanced the totally new concept that *a surface is a space in itself*.'" (*ibid.*, *emphasis in the original*)

Moreover, such way of tapping into spatial problems will also characterise the approach to the modelling of the spacetime developed by physicists during the 19th century[3].

"A Deleuzian multiplicity takes as its first defining feature these two traits of a manifold: its variable number of dimensions and, more importantly, the absence of a supplementary (higher) dimension imposing an extrinsic coordinatization and, hence, *an extrinsically defined unity."* (pp. 12–13, *emphasis in the original*)

Therefore, in light of these two main features, the idea of manifold is used as a way to dwell into the concept of multiplicity. Deleuze expresses this by clarifying that:

"Multiplicity must not designate a combination of the many and the one, but rather an organization to the many as such, which has no need whatsoever of unity in order to form a system" (Deleuze, 1994, p. 182)[4].

A central point about essences is that they do possess a defining unity and are taken to exist in a transcendent space that contains them or in which they are embedded. Instead, a multiplicity, "however, many dimensions it may have, … never has a supplementary dimension to that which transpires upon it. This alone makes it natural and immanent" (Deleuze & Guattari, 1987, p. 266).

To explain the ontological difference gained by replacing essences with multiplicities we need to clarify the way in which, in such a metaphorical language, multiplicities relate to physical processes that generate material objects and kinds.

DeLanda (2002) establishes relations "between the geometric properties of manifolds and the properties which define morphogenetic processes" (p. 13), mainly using the theory of dynamical systems that considers *manifolds as models for physical processes*.

To summarize this complex point in few passages:

- the dimensions of manifolds are used to represent properties of a particular physical process or system (each degree of freedom of an object is mapped into one of the dimensions of a manifold).
- A manifold is the *space of possible states* (*state space*) that the physical system can have.

- The *state of the object* at any given instant becomes a single point in the manifold.
- A curve or trajectory on a manifold, produced by the representative point moving in this abstract space, captures *changes of state*.
- The space state captures not static processes of objects but how the properties change, therefore it *captures a process*.

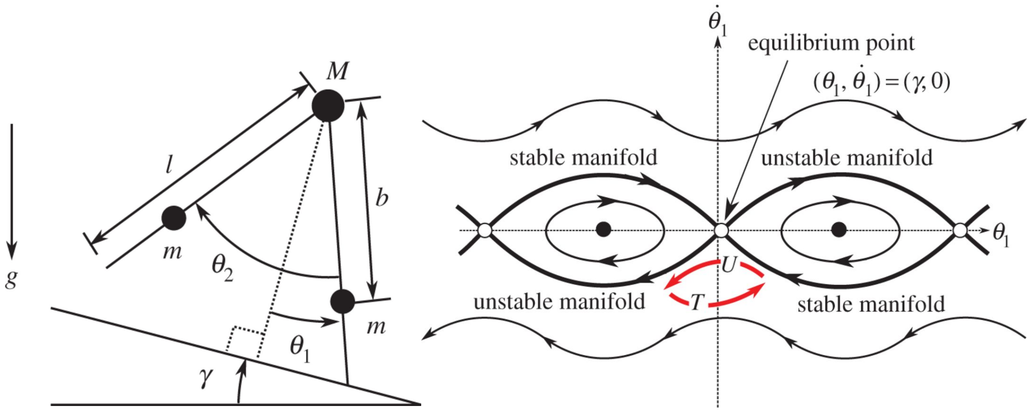

Figure 2.1. Compass-type passive dynamic walking model for the dynamical analysis of passive dynamic walking, and phase diagram of $(\theta_1, \dot{\theta}_1)$, which captures the changes in the angle of the stance leg θ_1 and shows the equilibrium point (saddle) around which stable walking occurs (retrieved from Obayashi, Aoi, Tsuchiya, & Kobubu, 2016)

Topological resources may be used to analyse certain features of these spaces in order to illustrate recurrent or typical behaviour common to many different models, and by extension, common to many physical processes. In particular, the concept of singularity is crucial, since singularities on a topological surface have an influence on trajectories, so they can influence the behaviour of the whole physical system by suggesting long-term tendencies of the system. Singularities might be attractors, topological points or limit cycles, which in turn attract trajectories to the same final state, to a steady state, or to an oscillatory state (see Figure 2.1 for an example).

Singularities are a crucial point in the definition of multiplicities, according to DeLanda, because of the way in which they lead to an entirely different way of viewing the genesis of physical forms. On the one side, "singularities, by defining long-term tendencies, *structure* the possibilities which make up state space, and by extension, structure the possibilities open to the physical process modelled by a state space" (DeLanda, 2002, p. 16, *my emphasis*). On the other side, singularities are recurrent and, in a sense, independent of the particular physical mechanism.

Multiplicities are, by design, *obscure and distinct*: the singularities that define a multiplicity come in sets, and these sets are not given all at once but are structured in such a way that they "progressively specify the nature of a multiplicity as they unfold following recurrent sequences" (p. 14).

This process, which characterises multiplicities, is referred to by Deleuze as *progressive differentiation*. How can the idea of progressive differentiation be translated into state-space terms? One singularity may undergo symmetry-breaking transitions and be converted into another one (e.g., from attractor to limit cycle): they are the so-called bifurcations. They are studied by means of control parameters with which one can display critical values at which the symmetry of the system is broken, and a bifurcation takes place. Much as attractors come in recurrent forms, bifurcations may define *recurrent sequences* of such forms. As an example, DeLanda describes the cascade that leads to the sequence conduction-convection-turbulence that occur in a water container when it is subjected to heat[5]. The example is useful because one can see that

> "a cascade of bifurcations may be faithfully realized in a physical system. This realization, however, bears no resemblance of the mathematical cascade. In particular, unlike the latter which is *mechanism-independent*, the physical realization involves specific mechanisms." (DeLanda, 2002, p. 21)

This *universality* (or mechanism independence) is a feature of multiplicities:

- "Unlike essences which are always abstract and general entities, multiplicities are *concrete universals*." (p. 22, *emphasis in the original*)
- "The universality of a multiplicity is typically *divergent*: the different realizations of a multiplicity bear no resemblance whatsoever to it and there is in principle no end to the set of potential divergent forms it may adopt." (*ibid.*, *emphasis in the original*)

This aspect is also consistent with a major difference with essences, namely the lack of resemblance, since multiplicities are responsible for the processes, but do not give form to their final products.

Another peculiarity of multiplicities is that, unlike essences, they cannot be sharply distinguished from one another, but must be thought as *meshed together into a continuum*, as they "combine the greatest power of being differentiated with an inability to be differentiated" (Deleuze, 1994, p. 187).

The idea of progressive differentiation can be explained using again an example that comes from the history of mathematics, namely the categorization of geometries by Felix Klein, which is known as his Erlanger Program (Klein, 1893). Klein succeeded in subsuming the various metric geometries under projective geometry. Instead of characterising geometries in terms of distinction among metric properties or in terms of the intrinsic properties of the objects, he categorized them *by their invariants under transformations,* that is, by means of external movements or the groups of symmetries the figures conform to. If we think of the geometries as forming levels of hierarchy (projective-affine-Euclidean), we can observe that the group of transformations embeds the group of transformations of the inferior level (and includes other transformations) and its theorems. *As a result, we can think of progressive differentiation in terms of transition among different geometries: moving up through the hierarchy of geometries means losing differentiation, while moving down we have more differentiated spaces (geometries).* In addition, if we think of expanding this spectrum, we can also include also topology and differential geometry in the discourse[6]. Topology as the least differentiated geometry is the geometry for which figures remain invariant under homeomorphisms, and under these transformations many figures that are completely distinct in Euclidean geometry can be deformed into one another (in a way, being the "same" figure, like the square and the circle). This also suggests that metric spaces arise from nonmetric spaces. In a similar manner, from an ontological standpoint, the cascade of broken symmetries gives birth to the metric space we inhabit from the non-metric continuum of multiplicities. In fact, the concept of metric/non-metric space can be compared to that of extensive/intensive.

To summarise: within the ontological model proposed by Deleuze, timeless categories that constitute the world populated by essences are replaced by historical processes, which are instead connected to material processes and exist in the real world.

Following DeLanda's description of multiplicities some main characteristics of *multiplicities as manifolds* have been highlighted. In brief, multiplicities (1) are defined by the distribution of singularities (that define tendencies in a process); and (2) they are defined by series of critical transitions which can take several such distributions embedded within one another and unfold them. The metaphorical language drives our discourse to a final grasp of multiplicity:

> “A multiplicity *is a nested set of vector fields related to each other by symmetry-breaking bifurcations, together with the distributions of attractors which define each of its embedded levels.*” (DeLanda, 2002, p. 32, *emphasis in the original*)

2.1.2 Multiplicity and the virtual

In which sense, then, the manifold metaphor should help us understand the concept of the virtual? The point is that *the status of multiplicities so defined is not actual nor possible but virtuality*.

In fact, we have discussed that the trajectories of points on the state space are sequences of possible states for the physical system, therefore they do possess the status of possibilities. Realising the possible, then, means that a specific history for the physical process takes place (along one of the possible trajectories). According to Deleuze (1990), “the regularities preside over the genesis” (p. 54) of the trajectories, which means that each of the possible trajectories is a consequence of the directions specified by the vector field, directions that in turn are to be ascribed to the singularities in a manifold. The vector field shows tendencies for the trajectories. The singularities define those tendencies by structuring the way in which points and curves might move around, but are never reached, so are never actualised. The ontological distinction between the trajectory in the phase portrait of a system and the vector field is explained by the mathematician Albert Lautman:

> “The geometrical interpretation of the theory of differential equations clearly places in evidence two absolutely distinct realities: there is the field of directions and the *topological accidents* which may suddenly crop up in it, as for example the existence of . . . singular points to which no direction has been attached; and there are the integral curves with the form they take on in the vicinity of the singularities of the field of directions . . . The existence and distribution of singularities are notions relative to the field of vectors defined by the differential equation. The form of the integral curves is relative to the solution of this equation. The two problems are assuredly complementary, since the nature of the singularities of the field is defined by the form of the curves in their vicinity. But it is no less true that the field of vectors on one hand and the integral curves on the other are *two essentially distinct mathematical realities.*” (Albert Lautman, quoted in Deleuze, 1990, p. 345, *emphasis in the original*)

In a sense, the virtual relates to something which gives structures to the space, but it does not get to be fully realised (actualised). The virtual “leaves behind traces of itself in the intensive processes it animates" (DeLanda, 2002, p. 44), and the multiplicity is to be thought of as immanent and progressively defining itself by differentiation: individuation processes link virtual multiplicities and actual structures.

For Deleuze (1994), the virtual and the actual are two dimensions of matter, which mutually presuppose each other. The virtual is immanent to matter—it does not transcend matter like some Platonic ideal form. The virtual is the dynamic indeterminism of matter, its *élan vital*. A first example, borrowed from Châtelet, will help to envision this aspect. He explains that the process of ice melting is not the possibility for the ice to melt, but that is really in the process of melting and turning into water: "water is of course 'potential' in ice, but above all it actualizes itself there." (Châtelet, 1993/2000, p. 19)

DeLanda goes further in the description of how intensive, virtual and actual are entangled in the immanence of multiplicities. Deepening the concept of virtuality also entails tapping into the concepts of virtual space and time, which should clarify the ways in which a continuum of multiplicities gets itself progressively defined through intensive processes. DeLanda discusses how time is responsible for the tension between the intensive and extensive, and the relations between non-actual temporality and actual history, explaining how the virtual is a part of reality, Nevertheless, exploiting this aspect would entail discussing in detail the way in which the intensive can engender the extensive, and it would take us too far. One example among others exemplifies the nature of virtual time. In the same way as the concept of Euclidean distance is rethought within the topological manifold, time has to be rethought. As Serres proposes, virtual time might resemble a 'crumpled handkerchief' rather than be thought of as linear, or as a sequence of consecutive time instants: this means that the distance between two far moments in time may collapse and these might be in close temporal proximity (Rotman, 2012).

One aspect which we have throughout highlighted is indeed the tremendous mobility that the virtual grants to reality. Rotman (2015) also stresses that the idea of the virtual is fundamentally grounded in mobility:

> "Virtuality concerns what is actualizable in an event, its potential, all the futures it could/might give rise to. Unlike the possible, which refers to determinations that are fixed but lack the conditions to realize them, the virtual is inseparable from tensions, problems, and open questions." (p. 5)

Such vision draws on the concept of virtual as it was further developed in mathematics by Châtelet, for whom a theory of virtuality fully explains the work of the mathematician with physico-mathematical concepts. In this section, we have traced some features that are ascribed to the virtual in DeLanda's reading of Deleuze. The concept of manifold has been used as a metaphor to dwell into the ontology of Deleuze and, in particular, in order to give a first glance to the concept of virtuality. The next section will analyse the main features that Châtelet grants to the virtual, with the aim of envisioning how the virtual is at play within mathematics. The two sections are seen one as the dual of the other, in the sense that the first introduces the virtual through a *mathematical metaphor*, while the second discusses the philosophical concept of the virtual *within mathematics*, that is, in relation to mathematical concepts. The two sections significantly complement each other.

The concept of the virtual does not emphasize the world as we know it but rather its potential to transform itself beyond its actual forms and configurations (Nivala, Salmi, & Sarjala, 2018). In my understanding of it, recognizing the virtuality of mathematical concepts means to trouble the commonly perceived 'definite image of abstract mathematics' and to look at the concept as something blurred, elastic and open to mobility. Moreover, instead of judging our vision in terms of more or less clarity or precision, engaging with the idea of virtuality means investigating what the concept might be or become.

2.2 The theory of virtuality in the work of Gilles Châtelet

In Chapter 1, I have already touched on the concept of virtuality as it is introduced by Châtelet in the context of his theory of physico-mathematical concepts. Now that our understanding of the virtual has been enriched by the mathematical metaphor of manifold, we continue to investigate meanings for the concept of virtuality by dwelling on the virtual nature of mathematical concepts, returning to Châtelet (1993/2000, 2010).

Rotman illustrates how Deleuze's and Châtelet's thoughts are intertwined. On the one side, according to Deleuze,

> "there are two poles of mathematical activity: what he terms the axiomatic, articulated here as the translation of mathematics into axiomatically based structures of sets; and the problematic pole, according to which mathematics is produced in response to problems (inside and outside mathematics) whose solutions account for the ontogenesis and character of these very structures." (Rotman, 2012, p. 255)

On the other side, for Châtelet,

> "diagrams coupled with gestures are the very means of ontogenesis, a principal strand in the becoming of mathematical ideas, objects and relations. Refusing the Aristotelean division between movable matter and immovable mathematics, Châtelet insists that mathematics can neither be divorced from 'sensible matter', from the movement and material agency of bodies, nor from the contemplative, a-logical and intuitive operation of thought; it combines them as 'embodied rumination'. He offers a material/corporeal account of mathematics, wherein gestures – which arise from 'disciplined distributions of mobility' of the body – are the physical vectors of mathematical thought." (p. 256)

In his "Figuring Space: Philosophy, Mathematics, Physics" (English translation of the original book "Les Enjeux du Mobile"), Châtelet discusses examples from the history of mathematics, recovering from the diagrams of such mathematicians as Oresme, Leibniz, Grassmann, Cauchy and Poisson the creative processes that led them to the formulation of new mathematical ideas. In brief, he discusses the creation/discovery of mathematical concepts by examining the role of diagramming and (imagined) gesturing around the diagrams and shows that the interplay between these two gives rise to new concepts. In chapter 1, we have already discussed the example of Oresme's configurationes to illustrate the force of diagrams, especially drawing on Châtelet interpretation that reveals the composition of the intensive and extensive in such powerful devices. In the following, I will characterise some features of the virtual (or potential) according to my reading of Châtelet and by drawing on some scholars, who examine and use Châtelet's work from the standpoint of philosophy, mathematics and mathematics education.

2.2.1 The virtual and the gestures/diagram interplay

Châtelet never gives a definition for the word "diagram". Sometimes he uses it to indicate a figure, or a doodle, a way of organising space in writing, but a diagram is also an experiment, a device and much more for him. In my opinion, it is not a viable or convenient decision to define here what a diagram is. For our purposes, what matters the most is to highlight how in his writings the interplay of gesture and diagram leads to a concept of virtuality in relation to mathematical concepts: diagrams are studied by Châtelet as a source of intuition, invention and discovery of mathematics *through the body* (Ng & Sinclair, 2018).

As we have already anticipated in the previous chapter, Châtelet considers the physical in the mathematical, rather than seeing the mathematical and the physical as separated. In so doing, he troubles the ontology of the relationship between mathematics and the physical world, as well as the classical vision of what it means to do mathematics. What is peculiar about this relationship is how the concept partakes in the virtual dimensions of the material world. The virtual is at play when we reconceive concepts less as static, abstract entities and more in terms of their power of affecting and being affected, their animating force, their potentiality and mobility, their capacity of giving rise to new configurations, alterations and mutations. So, for example, the circle can be thought of in terms of the virtual motions that it generates instead of being thought of as a static geometrical object. In "L'enchantement du virtuel", Châtelet (2010) takes the example of the circle when he discusses how points might be considered not as given in the plane, but as being somehow algebraic powers. For him, an 'abstract' point of the circle has absolutely no interest. Instead, one will have really said something interesting about the circle when one will "build functions on the circle or, for example, put some sine [curve]", or will "wrap a straight line in a circle with the sine, a constantly dynamic perspective in mathematics" (p. 6, my translation).

Following Châtelet, the relationship between gestures and diagrams can be rethought, through their coupling and looking at gestures as "capturing devices" and diagrams as "physico-mathematical entities". De Freitas and Sinclair (2014) notice the relevance of his vision with respect to present literature: "In contrast to current work on gestures, on the one hand, and diagrams, on the other hand, Châtelet insists that separating one from the other is both awkward and possibly misleading." (p. 64). For the philosopher-mathematician:

> "A diagram can transfix a gesture, bring it to rest, long before it curls up into a sign, which is why modern geometers and cosmologists like diagrams with their peremptory power of evocation. They capture gestures mid-flight; for those capable of attention, they are the moments where being is glimpsed smiling." (Châtelet, 1993/2000, p. 10)

Châtelet argues that the diagram is by its very nature never complete, and the gesture is never just the enactment of an intention. Instead, the two participate in each other's provisional ontology:

> "Like the metaphor, they [diagrams] leap out in order to create spaces and reduce gaps: they blossom with dotted lines in order to engulf images that were previously figured in thick lines. But unlike the metaphor, the diagram is never exhausted: if it immobilizes a gesture in order to set down an operation, it does so by sketching a gesture that then cuts out another." (*ibid.*)

Châtelet insists that gestures and diagrams are both pivotal sources of mathematical meaning and they mutually presuppose each other and share similar mobility and potentiality. De Freitas and Sinclair underline that "For Châtelet, diagrams 'lock' or 'capture' gestures. 'Capturing' is contrasted to 'representing' in that the latter is bound to a regime of signification that curtails our thinking about diagramming and gesturing *as events*." (de Freitas & Sinclair, 2014, p. 64). Briefly speaking, if the gestural gives rise to the very possibility of diagramming, so the diagrammatic gives rise to new possibilities for gesturing. Instead of being seen as external representations of existing knowledge, the diagrams are "kinematic capturing devices, mechanisms for direct sampling that cut up space and allude to new dimensions and new structures" (p. 65). The coupling and interplay of diagramming and gesturing are of interest for their being "embodied acts that constitute new relationships between the person doing the mathematics and the material world" (de Freitas & Sinclair, 2012, p. 134).

Rotman (2008) considers as an essential aspect of diagrams in Châtelet's perspective their "after-life", the future alterations that are latent in them and that never get "exhausted", suddenly bringing into life new unexpected gestures and movements. Thus, the diagram is a material surface, which is not an inert static part of the mathematical event but actively and dynamically partakes in mathematics thinking and learning, with its gaps and flaws. Mathematics emerges out of the actual and virtual mobility of the gestural and the diagrammatic, prompted by the continual movement and becoming that shape the activity. This means that the diagram which is created by the mathematician is thought of as something which is not detached from the reality and movement of the body, and which is capable of excavating between the actual, the sensible and the virtual.

Châtelet argues that the diagram is by its very nature never complete, and the gesture is never just the enactment of an intention. I understand the gesture/diagram interplay as a mutual co-provocation and mobility, which lies behind the creative act of the mathematician (and, by extension, the doer of mathematics) at work with diagrams through his/her own body.

In the foreword to "Figuring Space" (Châtelet, 1993/2000, p. xviii), Knoespel summarises the main features that Châtelet ascribes to diagrams:

- Diagrams constitute technologies that mediate between other technologies of writing.
- Diagrams create space for mathematical intuition.
- Diagrams are not static but project virtuality onto the space which they seek to represent.
- Diagrams represent a visual strategy for entailment.
- Diagrams are mediating vehicles which means that they cannot only be recovered but rediscovered.
- Diagrams have a pedagogical force that could be integrated into mathematical education.

A gesture, on the other hand:

- inaugurates a family of gestures, and gestures inaugurate dynasties of problems;
- awakens other gestures, contains references to the virtualities to which it alludes without reducing them;
- is elastic, mobile with the highest possible degree of freedom (movement);
- envelops before grasping something, sketches its unfolding long before denoting or exemplifying: "one is infused with the gesture before knowing it" (Châtelet, 1993/2000, p. 10);
- is not produced but rather "cuts out" forms in movement.

Diagrams and gestures provoke and capture virtuality. To put it differently, I imagine a double non-dialectic process of mutual composition of gesture and diagram, which conjures the movement of the body and the process of expressing thought as they are in becoming together; mathematical intuition, then, following de Freitas and Sinclair (2012) "is less about mystical insight into an ideal realm and more about the pre-linguistic apprehension of embodiment itself" (p. 138).

2.2.2 The virtual as horizon/hinge-horizon

For Châtelet, the idea of the virtual is grounded in the mobility of the body, as the mathematician gestures, moves and opens spaces in her exploration/invention process. Virtual embodies movement. "Virtuality is that which, in movement, permits to knot together an 'already' and a 'not yet'" (Châtelet, 1993/2000, p. 19). Gestures and diagrams that aim at capturing matter are not index of comprehension but are important since they capture ways in which a horizon of possibilities may be altered. Therefore, the horizon has not to be thought of as a barrier, but as the place of intuition. Knoespel uses the metaphor of a marble block, which is going to be sculpted by an artist: what really interests Châtelet, he says, is not the potential *of the block* before the statue, but the potentiality *of the space* that surrounds the block and the sculpture or the future alterations.

> "The horizon is neither a boundary marker that prohibits or solicits transgression, nor a barrier drawn in a dotted line across the sky. Once it has been decided, one always carries one's horizon away with one. This is the exasperating side of the horizon: corrosive like the visible, tenacious like a smell, compromising like touch, it does not dress things up with appearances, but impregnates everything that we are resolved to grasp" (p. 54)

Interestingly, we have seen that Châtelet's perspective on diagrams brings about the body in a significant way, the horizon of a diagram being populated by the gestures and the future alterations that spring from it. I found here some connections with the work developed in mathematics education by Nemirovsky and colleagues (2012). These researchers discuss the concept of horizon of a perceptuo-motor activity as a "realm of possibilities" within a phenomenological framework. They write:

> "These realms of possibilities cannot be fully determined because they include countless possibilities and do not comply with any single definition. But this lack of determination does not make them any less real." (Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012, p. 291)

Perceiving entails projecting an indefinite number of relations, perspectives, sensations that could be at play within little variations that could occur to us in that situation, or event. But the movement, which permeates the body, is all but relational. As Nemirovsky (2017) puts it in a more recent work:

> "What counts for a virtual are relations of all kinds; a cluster, for instance, becomes in relations of proximity, of ow and viscosity, of temperature and density distributions, and of distant electro-magnetic fields, among others. Virtuals are not amenable to full determination because the relations that count for them are open-ended, not yet set, and underspecified. […] The virtual is delimited not by defined boundaries, but as a horizon: *as we explore a virtuality, the horizon recedes, opening up hitherto-occluded regions.*" (pp. 254–255, *my emphasis*)

An illuminating example that Nemirovsky also puts forward proposes that thinking of the possible futures for a child embodies the concept of virtuality whether we understand the virtual as “not yet”. In envisioning this example, one is confronted with the fact that, despite the seeming contradiction, the negative determination of a thing, like the future of that child, still conditions and permeates the ways in which things become, or the ways in which one is in the world.

2.2.3 The virtual as allusion and enchant(e)ment

Châtelet’s style of writing is highly metaphorical, sometimes obscure and poetic at the same time. His style resonates with the features which we have sketched for the virtual until this moment: *elusiveness, mobility, obscurity*. While speaking about the role of the diagram, Châtelet explains:

> “Diagrams are in a degree the accomplices of poetic metaphor. But they are a little less impertinent - it is always possible to seek solace in the mundane plotting of their thick lines - and more faithful: they can prolong themselves into an operation which keeps them from becoming worn out. Like the metaphor, they leap out in order to create spaces and reduce gaps: they blossom with dotted lines in order to engulf images that were previously figured in thick lines. But unlike the metaphor the diagram is not exhausted.” (Châtelet, 1993/2000, p. 10)

So, the diagram (and, by extension, the virtual) acts within a metaphorical logic to allude to something different. An allusion “compresses and unfolds, but is not the same as either an abbreviation or an explanation” (p. 12). Another important feature of the virtual, which is described by Châtelet in relation to diagrams, is indeed its power to evocate without strictly defining, to put in motion without constraining movement along a specific path. Allusion is able to give enough space (and freedom) to intrinsic indeterminacy to unfold thought, since it carries enough ambiguity to allow thought experiments to take place. Moreover, the allusion in Châtelet’s work is accomplished with a poetical way of writing that carries with it a whole spectrum of sensation often detached from objective sciences: a sense of “enchantment” (from the French “enchantement”) to which Châtelet explicitly refers in his work (Châtelet, 1993/2000, 2010).

I understand the term enchantment in both its possible nuances of meanings. One is that which considers enchantment as fascination, or delight; the second alludes one to magic and charm. Virtuality carries both ways of sensing mathematical objects. Châtelet transmits with his words a profound fascination for the mastery of the mathematician-artisan, who excavates new mathematics from the blurred surface of the diagram by means of thought experiments. At the same time the mobility of the relations between physico-mathematical objects seems to perform a dance under our eyes, which is everything but magic, and brings with it the same genuine enthusiasm of a child in front of a magic trick.

2.3 The virtual in mathematics education

This section focuses on studies in mathematics education that have used Châtelet's work to dwell into the nature of mathematical activity. The aim of the section is to stress that there is vibrant research that considers the idea of virtuality as crucial for a deep understanding of mathematical activity and thinking, with specific emphasis on the role of the body in such processes, and to illustrate the contributes that this line of research brings forth.

There is a wide interest in research on the role and involvement of gestures in the teaching and learning of mathematics, as it is evidenced in the "Compendium for Research in Mathematics Education" (Cai, 2017), in which 14 out of 38 chapters mention or reference studies on gestures. Embodiment studies somehow traverse the classical branches in which mathematics teaching and learning are divided and have reached the status of an independent branch in some context. Nevertheless, differently from common research in mathematics education, in which gestures are studied as *evidence* of a person's conceptions about mathematics (e.g., Edwards, 2009), the works that rely on Châtelet's thought commonly investigate gestures in terms of their potential of actualising the virtual and in relation to diagrammatic activity, as a way of eliminating gaps between doing and thinking, or gesturing and scribbling. Although these studies have been already mentioned in the previous part of the chapter, in the following I want to highlight their specific contributions to this line of thought.

Sinclair, de Freitas and Ferrara (2013) use Châtelet's perspective on diagrams as site of inventiveness to create a new framework for attending at creative acts in the mathematics classroom, exploring the ways in which computer-based technologies might occasion the leaps into the virtual nature of mathematics. According to Sinclair and colleagues, a creative act:

> "1. introduces or catalyzes the new—quite literally, it brings forth or makes visible what was not present before,
>
> 2. is unusual in the sense that it must not align with current habits and norms of behavior,
>
> 3. is unexpected or unscripted, in other words, without prior determination or direct cause,
>
> 4. is without given content in that its meaning cannot be exhausted by existent meanings." (p. 242)

The first characteristic relates specifically to the process of actualising the virtual, and all the four aspects point "to the centrality of the body and its movement (actions)—rather than internal mental disposition—in creative acts" (*ibid.*).

This perspective on inventiveness consistently challenges existing frameworks in mathematics education that traditionally consider creativity as something which relies on the individual, identifying creative acts as occurring at the confluence of multiple agencies, and recognizing the social and material nature of creative acts. In line with Châtelet's vision, "the concept of the virtual becomes the animating force of the mathematical, giving *flesh and mobility* to what might have been otherwise considered abstract" (p. 252, *my emphasis*). The idea of a mobilized mathematics is further expanded by Sinclair and de Freitas (2014), as these authors propose the idea of a virtual curriculum. In fact, they offer an alternative design for classroom activities that, rather than focussing on the progression from concrete to abstract concepts, wants to rethink mathematical concepts and objects by recovering their virtuality. For example, the authors consider the triangle often introduced since the early grades as the concrete, rigid shape or sign, and ask: "How would the curriculum change if we rethink the triangles generated through transformations, stretched into differing shapes and sizes, in terms of this virtual mobility?" (p. 564). Mobilizing mathematical concepts entails rethinking pedagogical choices (how to approach some topics in the classroom) as well as curricular ones (that is, whether and when the students should encounter those topics during the unfolding of the curriculum). Sinclair and de Freitas mention other examples that come from Châtelet, like the rectangle as the mobile unit for multiplication (based on Grassmann's theory of extensions), which creates a new space engendering the movements of an origin point along two different directions. Another example which I consider worth mentioning here is that which discusses the pivotal role of the concept of zero by imagining a new kind of number line, one that emphasizes the generative role of zero and its way of creating "a symmetry of choices toward the positive or negative magnitudes" (Sinclair & de Freitas, 2014, p. 571).

Châtelet indeed explains that "The positive real numbers being given, I can obtain the negatives by mirror symmetry and then *add* the zero element. [...] I can also conceive the positive numbers as one of the sheets proceeding from a folding of a straight line; the point *O* then appears as a pivot point, and therefore as an *articulation*" (Châtelet, 1993/2000, pp. 94–95, *emphasis in the original*; Figure 2.2).

Figure 2.2. The creation of the number line as "spilling out" from 0 (Châtelet, 1993/2000, p. 95)

In another point of the book, Châtelet proposes another particular diagram, which conveys the idea that the point zero is opening out the number line into "two branches, into *two symmetrical mobile points"* (Figure 2.3):

> "Zero is the point where the first degree of generation is torn apart (that through 'mobile points'), and it is this which, once again, is going to orchestrate a disequilibrium in order to grasp in a single intuition two points going along a straight line in opposite directions. To master this opposition, we have to decide to see *two motions in one,* by mobilizing a contour which envelops the folding of the one on the other, and therefore to let oneself be dragged into a second dimension: splitting itself, the mobile point invites another dimension" (Châtelet, 1993/2000, p. 121, *emphasis in the original*).

Figure 2.3. A second diagram for the number line (Châtelet, 1993/2000, p. 121)

Sinclair and de Freitas notice that this new way of thinking about the number line recovers a new, creative dimension:

> "This 0 is the crotch of two fingers, the fulcrum of the teeter-totter. It is not the clichéd taking away of the last cookie because it requires the carving out of a new space that, once created, generates new mathematical objects (negative cookies!)" (Sinclair & de Freitas, 2014, p. 572).

This example is central for many reasons. First of all, it illustrates how intensity is the dimension through which the virtual operates, as opposed to extension, which is instead conjured by the line. It also exemplifies the virtual as an "indeterminate dimension" in matter that "quite literally destabilizes the rigidity of extension" (Châtelet 1993/2000, quoted in Sinclair & de Freitas, 2014, p. 573), and that evokes the idea of a fold, in the sense of Deleuze[7]. The example also involves the creation of a diagram, which is able to capture a gesture (namely, the bifurcation created by two stretched consecutive fingers) and, at the same time, alludes to potential mobility and to a new dimension. To summarise: "Virtuality can be thought of as a kind of intensity or potential energy that is embedded in that which is actualized in physical extension" (Sinclair and de Freitas, 2014, p. 573).

According to Roth & Maheux (2015a), Châtelet shows through the virtual

> "how thinking begins as a germ that is open to different trajectories to come. In concretizing itself, thinking develops. At the same time, undeveloped mathematical thinking is moving toward developed mathematical thinking so that in moving, thinking is transforming the process of thinking and not only the object of thinking." (p. 275)

In this work, the authors describe the activity of some mathematicians focussing on their diagramming: the flow of movement they look at is intended to involve the dynamism of mathematical thinking as well as the bodies of the mathematicians (see Intermezzo 2 for a wider discussion).

In another work (Roth & Maheux, 2015b), these authors draw on the work of Châtelet in light of pursuing a description of the morphogenesis of the figures (configurations) assembled by a student working with Tangram shapes. The concept of virtuality is used to grasp the invisible in mathematics education, with all its tensions and problematics, within a phenomenology of the inapparent.

In a study carried out in 2015 (Ferrari & Ferrara, 2018) we examined the role of diagrammatic activity in relation to tool use, using Châtelet's perspective on the gesture/diagram interplay. The students were engaged in graphing motion experiences in the context of a teaching experiment with the software WiiGraph. In particular, the focus was on the activity of making a circle and the study of its parametrical functions, as they were explored by means of the technology and then discussed through diagrammatic activity on a whiteboard. Telling the story from the point of view of the diagrams, which evolve by manipulations and gesturing of several students, the study examines the new dimensions and movements that arise from, within and about the working surface, as dynamic sources and sites of mathematical thinking. In so doing, it shows how tool use creates new and unexpected diagrams and new ways of envisioning and embodying gestures that mobilize those diagrams.

Ng & Sinclair (2018) investigate the use of 3D pens in the context of graph production and observe new forms of diagrammatic activity, which they interpret as new mathematical meanings that are both physical and abstract. Their study is of interest for our discourse about what graphs look and feel like using different tools or engaging with specific diagrams (see Chapter 1). In particular, the authors investigate the students' slowing down in drawing graphs of function with 3D pens, or new gestural forms of thinking about tangent to a curve that arise from the creations of "tangible" curves and tangent lines. The work of Châtelet allows these authors to grasp the dynamic interplay between the gestures and the diagrams-objects that are produced with the 3D pens.

Closing this section, I like to use once again Châtelet's words, when he writes:

> "Potential – the particular patience attached to each moving body – is exactly the thing that evades the clutches of an abstraction that seized mobility from, or granted mobility to, beings. The thought of the potential does not siphon mobility off from the motor to the moved, it does not pour it from a full receptacle into an empty one. The motor and the moved are not two inert beings opposite one another, transmitting a quality; the moved is not the only one to change: the motor possesses the form, but can only act in the presence of the moved. The moved is *awakened* to mobility; there is a whole preparation of the moving body to the superior form and, at any rate, not just any form could imprint just any matter; the ass cannot learn, but the pupil, even an ignorant one, can. That is, moreover, what is at stake in learning: to create a kind of tension that answers to the particular call of the pupil. To learn or teach, to accord or cede mobility gradually to a body, is always to invent a new homogeneity – a potential

– and to resist the expeditious processes of the 'transference of information'." (Châtelet, 1993/2000, p. 19, *emphasis in the original*)

In this quotation, Châtelet draws on two of the main aspects that are at the core of this dissertation, namely movement and learning. He advises us to consider the virtual (potential) and to avoid a mere transfer of information in teaching and learning mathematics. I close the chapter with a final metaphor for the virtual, which is thought of to recover the features of the virtual we have mentioned throughout these pages, and to open new questions to be investigated in the following.

2.4 A final metaphor for the virtual

The opening image of this chapter is a piece of art titled "Fold Number 1" (2017, acrylic on dibond, 44" x 46") by a contemporary artist, Katy Ann Gilmore. Katy Ann Gilmore is a visual artist living in Los Angeles; she received a BA in Mathematics, Art and Spanish. Some of her works caught my interest for they try to capture relationships between two and three dimensions. "Fold Number 1", for example, is a paper cut-out that, together with straight lines, creates the illusion of a folded surface; other works, equipped with curved lines, create space for the eye to imagine convex shape and profundity in the flat original support. Gilmore also makes wide—human-body sized—murals on walls that are 'gates' to new worlds, in which she uses straight lines that concur to the horizon line in innovative ways.

This piece of art supports my understanding of the virtual in Châtelet's terms:

- the picture literally shows a paper *cut-out* that uses perspective projections to get the illusion of a third dimension: *virtual cuts out space*.
- It is quite literally a folded piece of paper: the fold is a way of capturing the mobility of the intensive and its relationship with extensity. As it is for the virtual, there are new hidden possibilities; if I imagine stretching the surface and moving inside it, I will encounter something that now is not yet reachable.
- As is the case with this piece of paper, which has been cut out and decorated in such a way, and for which the horizon is set, once you see new dimensions, you can no longer ignore them: *there is no 'going back' from the virtual*.

- The lines inside the paper shape convey the sense of profundity, and these lines are often interrupted as if they wanted to generate the perception of a surface which is not completely smooth (or even) in all its parts. Eventually, the fact that the surface seems to be not evenly smooth may evocate the idea of a vibrant, open to mobility, support. As I have tried to highlight throughout the chapter, dealing with the virtual engages material activity and asks us a shift of attention to what something might look like or become, that is, to the vibrant horizon around things.

In addition, I do not interpret this piece of art as an optical illusion. I mean that what I see does not entirely depend on the focus of attention. This is what happens for example with the duck/rabbit used by Ludwig Wittgenstein in his "Philosophical Investigations" to describe two different ways of seeing: "seeing that" versus "seeing as", which, briefly speaking, may depend on which parts of the figure I first notice. In the case of Gilmore's cut-out, I do not see two things into one single drawing/form, rather, thanks to its design, I see a 3D object in this 2D figure, to which I add a new dimension (and perception).

This argument opens room for new significant lines of thought, like the one related to the question: How does the virtual connect to perception? This question resonates with a vision of perception which does not see attention as a sign of the relationship between me and a (mathematical) object; it rather wants to stress that, in every encounter, I am affected by, and affect, that materiality. The issue of 3-dimensionality is relevant here since I am conjecturing the addition of a new dimension to the original matter that is used to compose the piece of art, not because paper is glued and re-worked in 3D space, but because this addition already lies within the 2-dimensional piece of paper.

We can interpret all of this saying that we are not simply seeing with our eyes, but with hands, ears, materials and devices, extensions of our bodies: "the object is seen but the path is felt" (Spuybroek, 2016).

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[1] Image retrieved from: https://www.artsy.net/artwork/katy-ann-gilmore-fold-number-1.

[2] Manifolds have large use in many fields of mathematics and physics especially in the context of modelling physical systems. A Riemannian metric on a manifold allows distances and angles to be measured. Symplectic manifolds serve as the phase spaces in the Hamiltonian formalism of classical mechanics, while four-dimensional Lorentzian manifolds model spacetime in general relativity.

[3] Whether we take reification as a lens to describe the historical developments that led to the concept of manifold, we might say that it has implied a transition from an operational way of thinking about curves to a more structural mode of thinking about them by means of charts and mappings to Euclidean n-dimensional spaces. In my understanding of DeLanda's words, instead, such a historical reconstruction reveals how one can trace the concept in terms of how new resources that had been emerging historically became problem-solving resources in the hands of some mathematicians. In the previous chapter we have already spent some time discussing the genesis of mathematical concepts, therefore the reader may find the proposed way of tapping into such discourse coherent with what has been already presented.

[4] In another book by Deleuze and Guattari, we read:

> "Unity always operates in an empty dimension supplementary to that of the system considered (overcoding) … [But a] multiplicity never allows itself to be overcoded, never has available a supplementary dimension over and above its number of lines, that is, over and above the multiplicity of numbers attached to those lines." (Deleuze & Guattari, 1987, p. 9)

Unity is the marker of essences, while the indeterminate number of dimensions characterises the nature of multiplicity, as we will describe shortly.

[5] DeLanda shows many examples from thermodynamics that are useful to illustrate the practical contexts in which some of the illustrated processes occur. Nevertheless, I have decided here to focus just on the main ideas that come from the mathematical model of the manifold, as a bridge to the concept of the virtual; some of these examples are briefly exposed, like in this case, but many others can be found in DeLanda (2002). One example I like to mention is the process by which a fertilized egg becomes a specific organism (*embryogenesis*). Essentialist interpretations of this process consider the egg to be pre-formed in its structure, namely they see the egg already possessing a distinct nature. Most biologists have now agreed that differentiated structures emerge progressively during the development of the egg, that is, that the egg possesses an obscure yet distinct nature, not completely predetermined by the biochemical materials and genetic information that constitute it.

[6] Kline (1972) observes that Klein characterised differential geometry in terms of the group of transformations which leave the expression for ds^2 invariant.

[7] The idea of fold relates to the Deleuzian vision of concepts as open-ended and unexhaustive, non-exclusive and unlimited, exterior and infinite. Briefly speaking, for Deleuze, all of the universe is a process of folding and unfolding the outside. This process creates an interior that is not an inside grown autonomously from the outside world, but merely a doubling of the outside.

Intermezzo: Movement

> “the movement of life is specifically of becoming rather than being, of the incipience of renewal along a path rather than the extensivity of displacement in space” (Manning, 2009, p.72)

Bodily movement in mathematics education

Since the corporeal turn prompt by theories of embodied mathematics in the 2000s, studies that consider the role of the body in mathematics education research often discuss bodily movement in the classroom as a crucial resource for teaching and learning (Edwards, Ferrara, & Moore-Russo, 2014; Radford, 2013; Radford, Edwards, & Arzarello, 2009). Many of these studies often strive to code multimodal engagement—hand gesture, eye gaze, prosody in speech (high-low pitch), bodily posture, and so on—as a way to infer correspondences with particular cognitive stages, levels of understanding or steps in a learning trajectory. This is typical of constructivist or acquisitionist perspectives. However, the risk of associations of this kind is to see bodily engagement as a placeholder of cognitive schemas already existing in mind and, therefore, to fall into old body/mind splits, instead of thinking of the body and bodily activity as “genuinely constitutive of knowing” (Nemirovsky et al., 2013).

Of particular relevance is the concept of multimodality, which borrows from the cognitive sciences the idea that the functioning of the brain as sensory-motor system is multimodal. It brings forth the thesis that, in the act of knowing, different sensorial modalities—tactile, perceptual, kinaesthetic, etc.—concur to the development of cognitive processes.

In the last decade, researchers have offered ways to rethink the notion of embodiment through new perspectives that insist in dissolving any conceptual-perceptual cut or dualism. For example, Nemirovsky and colleagues (2013) take a non-dualistic stance on learning in informal settings to describe how perceptuomotor integration partakes in mathematical thinking about graphing motion, pursuing a phenomenology of lived experience. De Freitas and Sinclair (2014) propose the concept of assemblage within a new materialist perspective to address the issue of the body in learning in a wider sense, which also comprises the body of mathematics (see Intermezzo: Inclusive materialism). These authors want to extend thinking beyond the single individual effort and to show how it occurs as distributed through material encounters of human and non-human bodies. Enactivist researchers posit a shift in the way they consider enacted mathematical activity as knowing itself, in a dynamic process that involves the learner acting and immersed in the environment (Maheux & Proulx, 2015). Others explore the image of a growing-making mathematics (Roth, 2016), use theories on material phenomenology (Hwang & Roth, 2011) or extend the idea of sensuous cognition (Radford, 2013), in an attempt to investigate classroom situations arguing for monistic views of cognition.

This overview on recent theories of embodiment in mathematics education makes the following point apparent: there is growing interest in the dynamic nature, movement or flow, of the mathematical activity rather than in what the activity allows learners to achieve and the way it does so. To say it differently, attention is more and more shifted to the proper encounters of learners with mathematical concepts and to the relational entanglement of movement and thinking in these encounters. In my work, I pursue this line of flight, drawing attention to the way in which movement and thinking are contiguous and push each other forward in mathematics.

Movement and thinking

In a very recent work, Roth and Maheux (2015a) propose a "dynamic approach to mathematical thinking", addressing the issue of how we might exhibit mathematical thinking in movement in a way that learning and movement are not reduced to schemas. De Freitas and Ferrara (2015) take a similar, more philosophical, stance as they show that mathematical concepts themselves are mobile, but the most freedom of movement belongs to thought. The dynamic, mobile nature that is to be characterised in these studies belongs not just to the process of knowing or to the body but to thought, and to mathematics itself, in resonance with Châtelet's (1993/2000) view of the virtual dimension of mathematics (Chapter 2).

In particular, in my understanding, the aforementioned studies offer ways to pursue a non-representational vision of gesture and bodily movement. They also show a visceral interest in the way in which movement might be better characterised and studied in the context of classroom situations, in order to embrace the mathematical activity of students and teachers (as well as researchers) in its entire complexity and profundity.

I share these concerns as particular angles from which I can deepen *how movement and thinking sustain and build up each other in mathematical experiences that involve tools and materials*.

To this aim, the next chapter will dwell into the work of Sheets-Johnstone (2009, 2010, 2011), which is (mainly) dedicated to elucidating the nature of movement as the foundation of our conceptual life.

As Sheets-Johnstone herself states, in her first life, she was a dancer/choreographer, dance professor/scholar, while, in her second ongoing life, she has a Courtesy Professor appointment at the Department of Philosophy of the University of Oregon. Her studies range from dance to philosophy to evolutionary biology.

I first encountered the work of Maxine Sheets-Johnstone in 2014, when my Master thesis supervisor suggested me to read her article from 2009 titled "Animation: the fundamental, essential, and properly descriptive concept". In that paper, Sheets-Johnstone proposes, among other things, a critique of the concept (and word) embodiment. She suggests that *animation* is the crucial concept to understand our being alive in the world, as (human) beings that are kinaesthetically and perceptually attuned to the world. She elaborates on this notion as a way to turn upside-down the problem that the term embodiment is supposed to have solved, that of bridging the gap between body and mind in the constitution of concepts. Sheets-Johnstone proposes that concepts are grounded in the corporeal and qualitative realm of the body, a body which in the first place learns about itself and the world *in movement*.

Since then, I have been reading other books and papers she has written, and I clearly remember that since the beginning I was fascinated from her astonishing way of capturing movement with evocative language and considering it as a grounding principle both at individual and evolutionary scale.

In the next chapter, I will therefore draw on some theoretical aspects that ground her perspective on movement with the aim to take a perspective on *moving-thinking* that will help us examine the dynamic nature of mathematical activity of students, who work with graphical representations of spatio-temporal relationships. These aspects will be exploited in the following chapters. Before going on with theoretical commitments to movement and thinking drawing on the work of Sheets-Johnstone, I want to tell the reader something about my personal relationship and engagement with the concept of movement, or, one of the reasons *why movement matters*.

Walking in the university hall

During the period I spent in Manchester as visiting Ph.D. student at Manchester Metropolitan University, I was reading Sheets-Johnstone's book and having weekly conversations about my research project with prof. Ricardo Nemirovsky. The idea of focussing my research on movement already fascinated me, since I had chosen to work in mathematics classrooms with graphing motion technology, which involves ample bodily movement to be performed in order to produce certain collectively shared mathematical representations. The walks and arm movement of the students interacting with the technology in the classrooms I observed were video recorded, together with classroom discussions, as I will detail in the following (see Chapter 5). All these movements were captured somehow. Moreover, we know from the literature the relevance of perceptuo-motor activities in relation to the development of mathematical meanings (Chapters 1 and 4). But I was struggling with the idea of giving those movements the right space in my writing, to capture such ever-going movement in all its complexity and elusiveness. I felt so many tensions about the sense(s) I had about movement (as something complex, enigmatic, unseizable but still everywhere present) and the ways in which I felt (in)capable of giving voice to movements – the easiest way being that which describes movement as a simple change in position, which was too reductive to me; even with that only purpose in mind, how to describe a movement *fully* (whatever this means)?

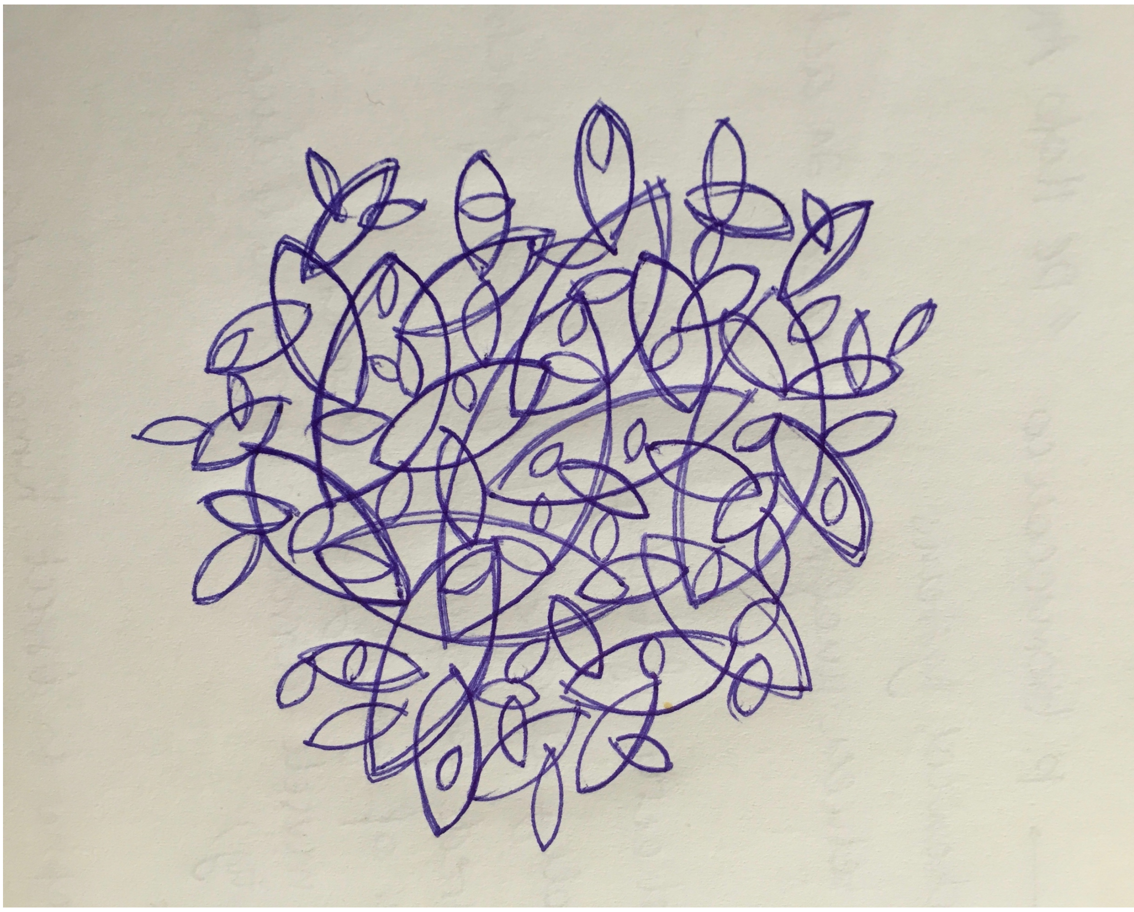
While I was drinking a coffee in the main hall of Brooks Building at MMU with all these ideas shaking, fighting and reverberating through me, mixed up with the difficulty of the first days of only-English conversations in a new country, I gazed towards the entrance, where droves of students were walking in and out. I saw so many styles of walking, filling up space by this continuous movement, in which each of the students participated for few seconds, entering my visual space, before disappearing in many directions. They were walking in and out, focussed in conversations with mates or rushing toward the lecture hall, with different pace and rhythm, and diverse permeating confidence and style. Many of them were walking with their smartphone in hand, gazing at it, texting or checking information, while chatting to a mate, and moving forward, stopping, gesturing around. Two main thoughts were evoked by this scene.

On the one side, what at first glance is a simple event, like ‘a student walking in or out of the university hall’, results from a mixture of many different nuances, ranging from tiny, imperceptible changes to wide and ample movements, combined with other actions, gestures, and so on, and making the separation of each element impossibly complex. Moreover, the entanglement of all these movements in concert, performed by many students getting to inhabit my visual space, was adding complexity but in a different degree, more like the composition of each path and performance than a hierarchical addition. It was conjuring a unique movement, even though heterogeneously composed.

On the other side, there is nothing like movement that conjures perceptions, feelings and thoughts at the same time: my being there being in some fashion to be partaking in that movement. And of course, all these thoughts were sustaining my understanding of movement as paradoxical, elusive and omnipresent event; once again, movement was going *beyond the words* I possessed to tell the story I was seeing.

I will ask the reader to keep memory of such *observation exercise*, since I think it is crucial to start thinking of movement from what commonly goes unnoticed, namely its elusiveness and omnipresence.

Summarising, in Chapter 2 we dwelled into the dynamic nature of mathematics, by means of the concept of virtuality, for which we have deepened the interest in movement within mathematics as a discipline and practice. In the next Chapter, a complementary perspective on movement will be developed, considerably exploding some of the issues that have been pinpointed in this Intermezzo and consistently involve the role of the (human) body in the picture.



3

Thinking in movement

This chapter unfolds the concept of movement from the standpoint of phenomenology as discussed by Maxine Sheets-Johnstone. Drawing on Sheets-Johnstone (2009, 2010, 2011, 2014, 2016), the aim of the chapter is to develop a discourse on movement that integrates cognitive, phenomenological and affective insights in order to open up a perspective in which the processes of moving and thinking are integrated and coherently sustain each other. In order to do this, I will first refer to the *primacy of movement*, namely why we should pay attention to movement and, more generally, why its importance gives us the possibility to ground a non-dualistic perspective on minds and bodies. Secondly, an in-depth discussion of the *qualitative structures of movement* will highlight the peculiar aspect of movement that emerges from a phenomenological analysis. Lastly, the exploration of the idea of *thinking in movement* will close the chapter. As in the previous chapters, I will not rely on a precise definition for the concept of movement, but I will continue to unfold possible meanings, drawing on the different contexts (biological, developmental, mathematical, …) I will touch upon.

3.1 The primacy of movement

Recalling Maurice Merleau-Ponty's essay "The primacy of Perception" (1964), Sheets-Johnstone speaks about the primacy of movement, namely its foundational character in terms of human development and animate forms' evolution. She states that movement constitutes the originating ground of knowledge and our sense-makings more generally:

> "Not only is our own perception of the world everywhere and always animated, but our movement is everywhere and always kinesthetically informed. The foundational significance of movement should in consequence be doubly apparent to anyone concerned to investigate the nature of animate life." (Sheets-Johnstone, 2011, p. 113)

The concept of movement is grounded in the idea of *animation*, which is a foundational notion that conjures aliveness, vibrant dynamics and the essence of life:

> "Animation is the ground floor of our being alive in all its affective, perceptual, cognitional, and imaginative guises, stages, practices, and surrounding worlds. In other words, animation grounds the full range of those intricate and varying dynamics that constitute and span the multiple dimensions of our livingness" (p. 467).

The concept of animation is borrowed from Husserl and from the widely acknowledged idea in the natural and human sciences that individuals, living bodies, are not to be considered in a vacuum, but exist in strict relation with the environment, or more precisely, they "are kinetically, affectively, thematically — *experientially* — anchored to and engaged in meaningful ways in a surrounding world, i.e. engaged in *synergies of meaningful movement*" (p. 13).

Therefore, by considering movement we can illuminate the way in which animate beings make sense of the world, as well as the epistemological foundation of learning to move oneself. A fascination with movement is shared with the scientific community, as it is evidenced from the many studies of infants (experimental studies or case observations; e.g. Bloom, 1993; Stern, 1985). In these studies, the scientific community puts forward evidence of how movement grounds conceptual understanding of containment, consequential relationships, weight and effort. Drawing on developmental studies on language that are grounded not only in words and discourse analysis, Sheets-Johnstone shows how a first and primary mode of thinking of the infant is *in movement*. To put it differently, humans discover themselves in movement and grow kinetically into their bodies:

> "Our capacity to make sense of ourselves, to grow kinetically into the bodies we are, is in other words the beginning of cognition. In making sense of the dynamic interplay of forces and configurations inherent in our on-going spontaneity of movement, we arrive at corporeal concepts. On the basis of these concepts, we forge fundamental understandings both of ourselves and the world." (p. 118)

Therefore, not only is movement the originating ground for understanding aliveness in general, but it is also the epistemological ground of our sense-makings and foundational for a repertoire of "I cans"[1] and for the development of a sense of ourselves as *agents*.

The primacy of movement, according to Sheets-Johnstone, has always been overlooked and this lack of attention is responsible for the perpetuation of the Western mind/body dichotomy. Such dichotomy emphasizes a Cartesian separation between what is bodily perceived, sensed and experienced and what instead is mentally acquired, conceptualised and thought of.

In Intermezzo: Movement, emphasis has been put on movement in the field of mathematics education, in research studies that try to overcome this dualism in relation to the grasp of mathematical thinking. In this section, I will briefly expose three lines along which Sheets-Johnstone articulates on this separation, as a way of showing the importance of movement in overcoming any split of this kind. The proposed examples come from other fields, which may appear far from the mathematical. The first line relies on the articulation of, and distinction between, analogical and symbolic thinking, the second line traits the mind/body problem first as a mind/brain problem and, finally, the last line presents the author's critique to the term *embodiment*.

3.1.1 Neanderthals: Analogical and symbolical thinking

At the beginning of the book, Sheets-Johnstone introduces the aforementioned dichotomy through the analysis of the controversy over the status of *Neanderthals* and *Homo Sapiens Sapiens*. She draws on the critique offered by the zoologist and geologist Stephen Jay Gould on the two main theses on Neandertals' relationships with modern humans: the first one (Stringer & Gamble, 1993) stresses that Homo Sapiens Sapiens has replaced Neandertal hominid without continuity between the two species, that is, modern humans arose out of Africa from a small population, which migrated first to Europe and then to all parts of the world. The second thesis (Trinkaus & Shipman, 1993) considers modern humans as they evolved from populations already spread across three continents in the form of *Homo Erectus*, and Neanderthals being their ancestors.

The first thesis is in particular sustained by the supposed distinction between the two species in terms of their mental differences (capacities). Stringer and Gamble (1993) describe a "change in behaviour" and especially a "symbolic behaviour", which apparently lead modern humans to establish campsite, settlements and new habitats. Symbolism is here defined as involving "mental substitution and appreciating association between people and, objects and contexts; once establishes, symbolism cannot be dropped or forgotten", while "symbolic behaviour requires memory and periodic renewal through repeated ritual" and that "[t]he objects used in such rituals tend to be standardized, leading to creation of a shared artform" (Stringer & Gamble, 1993, p. 203, quoted in Sheets-Johnstone, 2011, p. 8). This may be interpreted by saying that symbolic behaviour is generated by symbolic code that specifies certain mental substitutions. The researchers refer to symbolic behaviour as a "all or nothing situation". Therefore, one might ask: How did symbolic behaviour (or mental substitutions) originate? Did language (or art) arise one day full-blown from mouths (hands) of hominids? Did mental substitutions suddenly arise from an unconscious mental domain and just as rapidly instantiate in human beings a momentous new behaviour? Speaking of campsites as symbols, Stringer and Gamble contrast "symbolic behavior" with mere "survival behavior". Sheets-Johnstone argues that campsites themselves are not symbols, and nor are objects such as the tools connected with them. These constructions achieve symbolic status only on the basis of *being currently read as symbols*; that is, they are symbols only from the interpretive perspective of Stringer and Gamble – and others – who read them as symbols of intelligence: "Their attributions are conceptually muddled because they are projections of their evaluations and not descriptive of the things themselves" (Sheets-Johnstone, 2011, p. 10).

But, how does the mental result in being distinct, opposed, separated from the physical?

> "Although the ostensible concern is with behavior — the fabrication of hearths (thus campsites), the establishment of social networks (thus settlements), the expansion into new habitats (thus colonization) — behavior is conceived as merely a physical happening — a mere survival event. To be something more than a mere survival event, behavior must be regulated by behind the scene mental codes that have somehow arisen and become operative. *Then*, behavior becomes symbolic. But there is nothing actually grounding the epistemological

> connection; there is only the contiguous placement of two words: symbolic behavior." (pp. 10–11, *emphasis in the original*)

The model proposed by Stringer and Gamble might be summarised in the cascade that involves (1) a mental code that gives origin to (2) behaviour that lead to (3) a product. The distinction between Ancient and Modern Humans is then summarised in the capabilities of the latter of symbolic behaviour (mental functioning), which overcomes the survival behaviour (physical functioning) that characterises ancient humans. Opposing categories of behaviour such as "survival" and "symbolic" strengthen the old Western division between 'physical' and 'mental'. In Stringer and Gamble's analysis, Neanderthals were able to emulate but not to 'fully understand' and this also drives to differences in terms of the social, which is intended as the association between people, objects and contexts in Stringer and Gable's definition of symbolism and in what they consider symbols and significant behavioural changes.

Many anthropologists and philosophers concur that symbol is a social phenomenon; another fundamental aspect of symbol is its *representational power*. For Sheets-Johnstone, the referential dimension of symbols, say, how the idea that a symbol "stands for something else" arises, for Sheets-Johnstone (2011) has to be rooted: it needs to be "anchored in some form of reality as readily perceptible, that is, as open to immediate awareness, as the social reality of other individuals." (p. 13)

Two interlocking ideas are pivotal in this sense:

- *symbolisation is a form of analogical thinking*;
- *analogical thinking is foundationally structured in corporeal representation*.

Such an understanding of symbolisation offers an interpretation of mind and body not as separate/opposed entities, but as a whole, in the reality of "everyday creaturely life".

To build upon this vision, Sheets-Johnstone starts from showing that

> "there is ample evidence showing that corporeal representation is a biological matrix: in the everyday animal world, there is a fundamental disposition to represent meaning corporeally in the form of tactile-kinetic gestures. By the same token, there is a fundamental disposition to understand meaning corporeally." (p. 14)

This disposition is natural disposition towards iconicity and semanticity, i.e. "there is an iconic rather than arbitrary relationship between symbol and referent, and a built-in semantic dimension to living bodies that is evident both morphologically and behaviourally" (*ibid.*). Briefly speaking, Sheets-Johnstone observes that "animate bodies are semantic templates" or, equivalently, that "corporeal representation is a fundamental biological matrix. It is a primary mode of communication and symbolisation. Where meanings are *represented*, animate bodies represent them corporeally. […] [A]nimate bodies are a primary source of meaning" (p. 15, *emphasis in the original*). Corporeal representation is a fundamental node of multiple kinds of communication: "Whatever the particular referent, the symbolisation is conceptually played out corporeally, along the lines of the body" (p. 16).

This thought is articulated in two main points:

- Humans are not the only ones who are given to symbolising behaviours;
- The origin of symbolisation cannot be reduced to an on/off principle (like in Stringer and Gamble's work).

If we consider (and try to explain the origin of) tool making in the history of human beings, we might examine as an example the tool 'stone'. The main point here, according to Sheets-Johnstone is that "[...] stone tools are not symbols; they are stone tools. But they are stone tools that have been crafted on the model of the body, namely, teeth. They are thus analogues" (*ibid.*). The primary analogy between stones and teeth is about a structural correspondence, and is grounded in a tactile quality, which is common to both tooth and stone.

On the one side, the origin of stone tool-making demonstrates that analogical thinking is grounded in the tactile-kinaesthetic body (and, also, non-linguistic concepts – such as hardness – are not inferior to their linguistic relatives). On the other side, analogical thinking does not necessarily eventuate in the production of symbols: it is a fundamental form of thinking that generates understandings on the basis of bodily experiences.

Moreover, invoking symbolic code or theoretical acts about "mental substitutions" as operative in the production of tool is not accurate. This creates discontinuity with respect to symbolic behaviour and, to this extent, also to artefactual and symbolic aspects.

Using symbolic codes, learning rules, mental substitutions and associations does not solve the problem of specifying the whole process. Somehow, according to Sheets-Johnstone, we have to recognize that a faculty was already present. Namely, that faculty is

> "the power to think analogically, to perceive similarities in relationships, and to use the body as a semantic template. In short, if corporeal representation is the cornerstone of analogical thinking, and analogical thinking is the cornerstone of symbolization, [...] far from being a matter of newly operating symbolic codes, learning rules, mental substitutions, or associations, symbolization was an extension of an already extant biological matrix" (pp. 17–18).

In sum, "*symbolization is latent in analogical thinking and analogical thinking is latent in corporeal representation*" (p. 17, *my emphasis*). Within this discussion on the status of Neanderthals, Sheets-Johnstone makes the point that when the basic biological matrix of corporeal representation is ignored, the mental takes the lead, therefore the mind/body dichotomy is strengthened and perpetuated. Whether we take for granted such assumption, we fall into the reiteration of the mind-body dichotomy. The discussion might have led us far, but it is significant to notice that the issue is subtler than one might expect and is implicated in the ways we speak about processes that concern humans and human bodies. Of course, it is a challenge to overcome such mechanism. But this brings us to the primacy of movement again, since it is the key concept to employ:

> "The challenge in articulating kinetic possibilities and dispositions is precisely to show how dynamic elements of movement and the tactile-kinesthetic body play out conceptually, i.e. analogically, in a way similar to the way in which stone tools play out conceptually both the tactile character of teeth and the spatio-kinetic character of animate form." (p. 32)

This in turn means that we should ground our understanding of movement on a different conceptualization of it, as we will detail in the rest of chapter, drawing on the profound work of Sheets-Johnstone.

3.1.2 Cartesianism

As a first point, a wider conception of movement requires that we go beyond a definition of movement that relies on a vision of movement as transport. According to Descartes, in fact, motion can be defined as a mere transfer in position: "the transfer of one piece of matter, or one body, from the vicinity of the other bodies which are in immediate contact with it, and which are regarded as being at rest, to the vicinity of other bodies" (Descartes, 1644, quoted in Sheets-Johnstone, 2011, p. 400). A Cartesian perspective on movement nullifies any sense of its dynamic. Focus is indeed put on matter, which is the object of the transport process: movement is reduced to the *positional* and the *happenstance*. In some sense, life is interpreted in terms of a sequence of still instances, which is a denial of a dynamic sense of movement, and matter is detached from mind, one way of interpreting the dualism introduced by Cartesian rationalism.

In addition, along such first line of thought, as a speculative argument within the issue of Cartesian dichotomy, the Brain-in-Vat (BIV) thought experiment is discussed throughout the book as a way of deconstructing the Cartesian separation of mind and body. The BIV argument concerns the imagined situation that we, as humans, are simply brains immersed in a liquid and connected to a computer that provide electrical stimuli that sustain brain activity. In such context, the computer would simulate the reality and experience of the external world and the person would be perceiving herself living in the same way as embodied brain would do. In philosophical terms, on the one side, this relates to the impossibility of deciding whether we actually are BIV or not, and in turn of going closer to the idea of knowledge, reality, mind or meaning. On the other side, and this is the point which is of interest here, the experiment puts forward the idea that we, as humans, only depend on our brain to be and live, while the body is only an accessory to the mind. Or, in other words, minds think (through the brain), while bodies *do*.

As a vantage point from which to tackle the issue of Cartesianism, Sheets-Johnstone discusses the question: "What is like to be a brain?"[2]. Tentative answers to such question result in the impossibility of grasping what the brain activity is really like. The author proposes to recognize the fact that primarily "to be a brain *is to be active*; *it is to be something other than simply an inert piece of matter*" (Sheets-Johnstone, 2011, p. 395, emphasis in the original).

Sheets-Johnstone draws on the thesis of the psychologist Roger Sperry that the brain is first of all *an organ of and for movement*. Hard-drive conceptions of brain fall short in taking into account this fundamental facet, namely, the "*kinetic character of brain matter*" (p. 417, *my emphasis*).

> "A Galilean-Cartesian construal of motion shifts attention away from the kinds of fundamental concern Aristotle had about movement toward not only mathematical concerns but inertial ones. Rather than taking movement as something to be understood and explained in its own right, a Galilean-Cartesian construal makes it a simply possible condition of matter" (p. 400).

The key element is again movement: in particular, in the following sections we will discuss how a comprehensive account of movement does consider it (1) as change itself, (2) as our natural way of being a body and (3) as our mother tongue.

3.1.3 Embodiment: Proprioception and kinaesthesia

In her 2009's article "Animation: the fundamental, essential, and properly descriptive concept", Sheets-Johnstone offers a fundamental critique to the notion of embodiment.

She considers the term embodiment a "lexical band-aid", which has been placed to cover the profound wound inflicted by the Cartesian split.

> "The term embodiment and all its derivatives are in truth linguistic embalmers. Instead of conceptually enlivening what they qualify—emotions, actions, subjectivity, experience, metaphor, conversation, perception, and so on—they conceptually embalm it, dressing it up in fashionable garb, i.e., garb that makes it look as if what they qualify is a living phenomenon, part and parcel of something right here and now in the flesh" (Sheets-Johnstone, 2009, p. 397).

This criticism of the embodiment turn is in line with the theoretical approaches of mathematics education research that have been discussed in Intermezzo: Movement, for which the bodily and the perceptual need to be reconsidered.

Following the history of proprioception, which is the beginning of the life of animate forms, it is possible to highlight how animate forms are attuned to movement and a sense of movement constitutes their connection to the world and their ability to be responsive to it. When it comes to humans, we can analyse this aspect by considering kinaesthesia and proprioception:

> "kinesthesia and the broader term 'proprioception' cannot be transmogrified into forms of 'action' or 'embodiment', or into a motorology and in any way retain their essential phenomenological qualities, qualities foundational to animate life. Indeed, tactile-kinesthetic invariants ground our basic species-specific human repertoire of movement possibilities and undergird our affective social understandings." (p. 396)

From a biological point of view, dedicated sensory systems make the human body unique as an object of self-perception and as a perceiving organism. Interoceptors located in organs and soft tissues monitor visceral functions and provide awareness of changes in the state of the body itself. Proprioceptors located in the muscles, joints, and vestibular apparatus provide information about the body's position and movement in space. Exteroceptors receive stimuli from the external environment and include the senses of vision, hearing, smell, taste, and touch. Kinaesthesia, the sense of movement, draws upon many of these receptors. Consequently, the perception of movement is multimodal. It integrates the far and near senses, providing us with awareness of the position and motion of our own bodies and the motion of other organisms and objects in the environment. Kinaesthesia as a sense "provides us with a substratum of knowledge of the body's position and posture, as well as knowledge of the movement of our limbs" (Moore & Yamamoto, 2012, p. 14). Thanks to kinaesthetic engagement we know where our body is, and we feel the direction of our movement, independently from our sight. What is significant about kinaesthesia is not only the identification with a sixth sense, or sense of movement (as discussed in Berthoz, 1997), but the fact that, from a physiological perspective, there are several mechanisms that involve receptors, exteroceptors and proprioceptors that concur to this unifying sense or perception of movement. As an example, muscle spindles (a specialized type of muscle fibres that is interspersed among the fibres in most muscles of the body) are responsible for sending signals that quickly message the muscles' changes in length and their speed of change to the central nervous system. Golgi tendons organs – at the junction of muscle and tendons – signal variations in tension, by measuring the force that a muscle exerts in the bone to which it is attached. Movement sensitivity is also related to the vestibular apparatus (in the inner ear), which interacts with gravity to transmit displacement to the central nervous system and is responsible for balance and spatial orientation as the body moves or stands still (in the absence of variations). Proprioception, as wells as touch, involves the perception about self and the neighbouring environment. For when I touch something with a finger, I sense the thing I touched, but I am also touched by the thing itself, and a deeper perception of the part of the body (e.g., my finger) which is in contact with the thing surface is elicited. About proprioception, far senses, like hearing and vision, do partake in the extension of our perception of movement. In brief, intensity and changes in sounds inform our perception of movement, and this is commonly experienced in our everyday life. Many research studies have highlighted that hearing plays a functional role in the perception of movement as it permits to detect the location of a sound's source and its movement away/towards us. It also has a social role, because the capacity to synchronize movement with spoken rhythms is crucial for human interactions (Moore & Yamamoto, 2012). Vision is also central to the perception of movement; however, despite its being one of the most studied senses, it is complex to understand its interrelation with movement (Cutting, 1997; To, Regan, Wood, & Mollon, 2011). The issue is quite intricated, and the discussion goes beyond the interests of this section. What really is relevant here is the fact that "the perception of motion depends upon a dense interrelationship of proprioceptive and exteroceptive signals, along with a brain somehow capable of melding, comparing, and interpreting sensorimotor information of many kinds" (Moore & Yamamoto, 2012, p. 19).

Central to the awareness of our own body movements are the proprioceptive sensors. These provide a constant, subliminal knowledge of the arrangement and motion of body parts. This awareness is enhanced by the sense of touch, in which contact and pressure receptors provide additional information about body position. Sheets-Johnstone further argues about the relationships between body movement and position and examines the experience of self-movement to highlight that it is a three-dimensional happening: movement is at the same time an *internal and an external phenomenon* (Kelso, 2009; Sheets-Johnstone, 2016)

Proprioception and kinaesthesia are therefore crucial elements for the understanding of movement and pivotal for a better understanding of cognition. Moreover, we have throughout described how kinaesthesia is not a *positional* sense but a *movement sense*, the experience of which constitutes a specific qualitative dynamic.

3.2 Primary qualitative structures of movement

> "Quality is what Galileo left behind. It is what Western science leaves behind, quality not only in the sense of kinetic quality, of course, but in the sense of sensory qualities generally. Quality is obviously less substantial than *objects*. Moreover, kinetic quality in particular is processual rather than substantive." (Sheets-Johnstone, 2011, p. 133, *emphasis in the original*)

In order to disclose the concept of movement from a qualitative standpoint, which has been emerging as a central issue in the previous section, I will touch on the methodological practice of free variation, which is used both by 19th-century physicist-physiologist Hermann von Helmholtz in the context of axiomatic geometry and by 20th-century philosopher Edmund Husserl as part of its phenomenological methodology. Sheets-Johnstone notices that there is a great affinity of thought between Husserl and von Helmholtz concerning the recognition of the high significance of movement in perception and the use of the methodology of free variation. We will discuss how performing free variations on a sequence of movements shows that an overall quality is preserved. A movement has a distinctive felt qualitative character coincident with a variation, which is at the core of our understanding of movement fundamentally as *change*.

3.2.1 Free variations

> "In phenomenology, free variations are a means of arriving at eidetic truths, that is, truths about the essential nature of the thing in question — perception, memory, willing, disliking, or whatever. One performs free variations by running through possible instances of whatever it is one is investigating. One thereby discovers what is essential to it." (Sheets-Johnstone, 2011, p. 169)

The methodological practice of free variation, which entails imagining the possible, and a tension between potential and actual freely-varied movement, was used by von Helmholtz (1971) with the axioms of geometry to discuss whether they are "necessities of thought". Despite he was not a phenomenologist, he used such methodology to give insights on mathematical knowledge. In so doing, he presents a thought experiment in which he proposes to perform free variations on a particular spatial theme, namely one which involves a flatlander[3] trying to grasp an object:

> "Let us, as we logically may, suppose reasoning beings of only two dimensions to live and move on the surface of some solid body. We shall assume that they have not the power of perceiving anything outside this surface, but that upon it they have perceptions similar to ours. If such beings worked out a geometry, they would of course assign only two dimensions to their space. They would ascertain that a point in moving describes a line and that a line in moving describes a surface. But they could as little represent to themselves what further spatial construction would be generated by a surface moving, as we can represent what would be generated by a solid moving out of the space we know. By the much abused expression to represent or to be able to think how something happens I understand — and I do not see how anything else can be understood by it without loss of all meaning — *the power of imagining the whole series of sensible impressions that would be had in such a case*." (von Helmholtz, 1971, quoted in Sheets-Johnstone, 2011, p. 173, *my emphasis*)

Therefore, it is through imagining the possible that von Helmholtz proposes that one moves from the world we know and inhabit to the world of possible experiences (cf.

§2.2.2). By combining the investigation of possible experiences through free variations (the *power of imagining*) and actual (active) experiences of movement (or, in other words, experiences of self-movement) he shed light on "*how we come to perceive*" (Sheets-Johnstone, 2011, p. 175, *emphasis in the original*). For both von Helmholtz and Husserl, freely-varied movement is a starting point for making inductive inferences – which are crucial to our perceptual experiences of objects – and, at the same time, it is an essential component of the power to perceive.

3.2.2 Tensional, linear, amplitudinal and projectional: The four primary qualitative structures of movement

Sheets-Johnstone states that the methodology of kinetic free variations *disclose four primary qualitative structures of movement*. These are not pre-existent the experience of moving, but get to be revealed only in movement, and are analytically separable only after the fact. Nevertheless, "experientially, they are all of a piece in the global qualitatively felt dynamic phenomenon of self-movement" (p. 123), that is, we cannot experience them separately, rather they are scattered throughout the process. This is one of the reasons why it is indeed very difficult to describe the qualities and the overall experience of movement[4].

The four cardinal qualitative structures are given by *tensional*, *linear*, *amplitudinal* and *projectional* aspects of movement. They are qualitative aspects that together constitute the dynamics of movement and relate to force or effort, space and time:

> "In a very general sense, the felt *tensional* quality has to do with our sense of effort; the *linear* quality with both the felt linear contour of our moving body and the linear paths we sense ourselves describing in the process of moving; the *amplitudinal* quality with both the felt expansiveness or contractiveness of our moving body and the spatial extensiveness or constrictedness of our movement; the felt *projectional* quality with the way in which we release force or energy." (*ibid.*, *my emphasis*)

Linear and amplitudinal qualities of movement are related to the spatial aspects of movement, since they capture the direction and extension in space of a movement, whereas tensional and projectional qualities are related to the temporal aspects of movement, since their combination is responsible for the intensive expression of a movement. Recovering the metaphor of Chapter 2, there is a manifold of possibilities regarding the temporality/effort/space of a movement, which makes movement as a concept a complex multiplicity to grasp and study; at the same time, despite its generative and dynamic character, movement presents itself with a profound unity among expression, nature and experience.

Grasping the qualities of a movement, we might also note that "we formally *create space* in the process of moving; we qualitatively create a certain spatial character by the very nature of our movement — a large, open space, or a tight, resistant space, for example" (p. 124, *my emphasis*).

I propose the reader to explore these four structures by performing free variations of a movement sequence, in a similar way with respect to what is proposed by the author in the book, but with a slightly different focus[5], for the sake of scope.

Imagine yourself in a corridor of your house. You are standing on your feet; your hands and arms are relaxed on the sides of your body. You start walking: as your right leg moves forward, then the left follows and your arms dangle slightly, for a few steps. We next perform free variations on this walking movement. For example, you can walk quickly, or change your speed as you go forth, gradually accelerating or brutally changing rhythm, and all these aspects are instances in a manifold of possibilities for the temporality of this (and any) movement. There is also a manifold of possibilities regarding the tensional aspects: you can move powerfully, with great tension in your steps; you can clump down the corridor; you can play around with the intensities of your movement, alternating or modulating them as you go forward. You can change the ways in which to project force: you can lift the right leg with initial great force, and leave the foot touching the ground without control; or you can perform the sequence of steps in a sustained but constant manner. You can also initiate your movement by projecting your head forward, while the rest of the body moves after (as if you were losing balance); or you can shift forward as if your pelvis was initiating and guiding the step, while the torso is dragged along by it thereafter. You can similarly vary the movement spatially, in both a linear and amplitudinal sense. You can emphasize the rotatory movement of your legs or zigzag in the corridor; you can make big steps or little ones and augment or restrict the way in which the arms partake in the whole movement, changing your arms' swing.

The example, a simple walk in a corridor, shows that there is an entire bundle of possible dynamic variations that can be described turning attention to the four qualitative structure we have underscored.

> "The question is, what is invariantly there through all these variations — and any further ones anyone could possibly imagine? What is invariantly there is in each case an overall *quality*. Whatever the variation, the movement has a distinctive felt qualitative character coincident with that variation, a felt physiognomic aspect which is in fact a constellation of qualitative aspects. These qualitative aspects — dynamic structures inherent in movement — enter into and define our global qualitative sense of any particular movement variation; they make all of the variations immediately distinctive to us *as variations*." (Sheets-Johnstone, 2011, pp. 122–123, *emphasis in the original*)

Through the example we also start notice that kinaesthetic experience cannot be reduced to a mere change in position, but, most fundamentally, is a matter of *change*.

> "Force, effort, or energy is continuously created in the process of moving; it is part of the global kinetic dynamic, the changing, shifting interplay of created spatialities and temporalities. Clearly, the gap between the experiential and the linguistic is not easily bridged, but kinetic experience is not on that account doubtful in the least. While fine-grained kinetic terms to describe the created qualities of movement are hard to come by — if not at times seemingly altogether lacking — the qualitative experience itself is kinetically unmistakable. When we pay attention to our own movement, we find that that non-verbal experience has a distinctive spatio-temporal dynamic coincident with the manner in which we are moving." (p. 127)

Movement is inherently spatial: we create space in the process of moving (ourselves), through two interlocked processes of (1) *kinaesthetically feeling a certain qualitative spatial dynamic* and (2) *kinaesthetically perceiving the three-dimensionality of the movement* (Sheets-Johnstone, 2010). Both processes concur to create a kinaesthetic-kinetic experience, namely the experience of a qualitative spatial dynamics and the three-dimensionality of movement. Movement is the qualities that it engenders. This is why we *do not have sensations but feelings or perceptions of movement*, "precisely in terms of our double spatial sense of movement: we 'perceive' our movement as a three-dimensional happening; we 'feel' the qualitative dynamics of our movement" (Sheets-Johnstone, 2010, p. 116).

The cardinal structures that constitute the movement create qualitative aspects that are usually part of our everyday experience of moving. What is habitual in a movement usually gets to be unnoticed. For example, whether we go down along the stairs and we step forward without finding a last descending step, our foot might touch the ground in an emphasized manner, breaking the expectation of the whole body and its preparation to that last step. We feel like we are falling, and the whole temporality of our movement is qualitatively different from what it would have been had we encountered a last step.

3.2.3 Movement and affectivity

As it is largely evidenced in the study of infants, we come into the world moving, and movement constitutes our first sensibility to the world. This opens room for the relationships between movement and affect to emerge. In particular, the work of infant/child psychologist Daniel Stern (1985) stresses a coincidence rather than a derivation of affect from kinaesthetic experiences. He proposes that affect should be "better captured by dynamic, kinetic terms" instead of using feelings ones, because affective states do have origin in the tactile-kinaesthetic body (Stern, 1985, quoted in Sheets-Johnstone, 2011, p. 74).

Drawing on Stern's work and on a long tradition of phenomenological studies, Sheets-Johnstone (2009) describes *affectivity* as the fundamental "responsivity" of life. Thus, affectivity characterises the way that bodily activity is implicated in collective feelings, or how bodies turn away or lean in and, at the same time, how they join with other bodies in coordinated movements. Animate forms of life enjoy (for good or bad) a congruency between affect and bodily motion, precisely because affect is lived through bodily movement. The dynamics of feelings (of comfort, agony, excitement, ...) coincide with micro-facial expressions, minute changes in bodily posture, foot-tapping rhythms, changes in heart rate, etc. Sheets-Johnstone posits that "the affective and the kinetic are clearly dynamically congruent; emotion and movement coincide" (p. 377). For her, emotions are not enacted, but *emerge in movement*. Enactivism, she suggests, falls short of recognizing this powerful "spatio-temporal-energic" dynamism that saturates all activity. Moreover, it fails to grasp the dynamically congruent relationship between affect, movement, and concept. She is at pains to show how emotions are not only "coping mechanisms" that evaluate or appraise or cope with the sudden break-down of rational discernment. She critiques the early systems theorists such as Varela (Varela, 1999; Varela & Depraz, 2005), who treat emotions as such when they study them only as responses to something not working or to surprise.

In avoiding the term 'enactment', we too want to resist the tendency to define emotion as "a movement outward". This way of thinking about emotion has perhaps fuelled theories of embodiment that treat bodily movement as the *externalization* of inner immaterial feelings[6]. Contesting this approach, Sheets-Johnstone points out that etymologically the word 'emotion' first signified the migration of peoples and geological transformations, and only in the eighteenth century took on the psychological flavour of "agitations or stirrings of mind, feeling, passion" (OED[7]). She emphasizes the earlier meanings to argue that emotions *are themselves motion* and do not connote motion in some indirect fashion, where one represents the other. She puts it concisely: "emotions move through the body at the same time that they move us to move" (Sheets-Johnstone, 2009, p. 379). Emotions do not *motivate* motion, as though some distinct interior force, but they do *inform* motion "every step, turn, gesture, clenching or quivering of the way" (*ibid*.).

The shuddering, trembling, quaking, constriction and heaviness that we feel at certain times are the thoroughly corporeal happenings of anger, fear, joy, anticipation, and so on. Therefore, emotions are not states but *moving phenomena*, because of a "natural binding of affective and tactile-kinesthetic bodies" (Sheets-Johnstone, 2012, p. 399). Accordingly, Sheets-Johnstone suggests that feelings of fear are "dynamically congruent" to kinaesthetic feelings of running away, while feelings of joy would be dynamically congruent to kinaesthetic feelings of moving towards. As we note in de Freitas et al. (2018), we should be aware that this aspect of congruency has to be problematised, since the same feeling may differently move different bodies, in different cultures or contexts. We cannot and should not claim that any movement is felt the same way by all people or on all occasions. When Sheets-Johnstone (2009) describes the affective kinetic dynamics of *joy* as that which "spatially expand the body outward and infuse it in a lightness and buoyancy that are spatially and temporally open-ended" (p. 395), we are left to wonder how she addresses the fact that such an expansive movement is joyful in certain cultures and not others. This problem needs to be addressed, in that events are always populated by multiple bodies with varying agencies. Incongruencies, tensions and ruptures fuel the heterogeneity of affectivity where a multiplicity of bodies (and agencies) is involved.

3.3 Thinking in movement

This section examines the idea of "thinking in movement" as it is proposed by Sheets-Johnstone and aims to consider the phenomenological integration of the two processes of thinking and moving as a starting point for opening new questions.

Sheets-Johnstone elucidates the experience of an improvisational dance, which is chosen and then described by the author as a paradigmatic example of thinking in movement. An improvisational dance is a unique event. In being a singular performance, there is no other dance to be compared with, nor past or future performances to be linked to, not even a plan to be followed: it "exists only in the here and now of its creation" (Sheets-Johnstone 2011, p. 420). In addition, the process of creating the dance *is* the dance itself. It is a fully generative, creative, dynamic process. In the same way as improvisation is process through and through, thinking in movement is motional through and through:

> "To say that the dancer is thinking in movement does not mean that the dancer is thinking *by means of* movement or that her/his thoughts are *being transcribed into* movement. To think is first of all to be caught up in a dynamic flow; thinking is itself, by its very nature, kinetic. It moves forward, backward, digressively, quickly, slowly, narrowly, suddenly, hesitantly, blindly, confusedly, penetratingly. What is distinctive about thinking in movement is not that the flow of thought is kinetic, but that the thought itself is. It is motional through and through; at once spatial, temporal, dynamic." (p. 421; *emphasis in the original*)

To elaborate on what it means (*how it is like*) to think in movement, the author assumes the perspective of the dancer engaged in the process of a dance improvisation and describes the process "from the inside" through a phenomenological account of the experience. The first point in this 'movement-thought experiment' is that improvisation means for the dancer to explore the world in movement. On the one side, this implies that "[q]ualities and presence are enfolded into [her] own ongoing kinetic presence and quality" (p. 422), engaging her directly with the here and now, without any gap between the "global dynamic world" which is perceived and "the kinetic world" in which she is moving. On the other side, the range of possibilities that are at stake for movement and the choices that create the dance as it is are not explicit. The world that the dancer is exploring in movement cannot be separated by the world she is creating in movement: "the idea that thinking is separate from its expression — a thought in one's head, so to speak, existing always prior to its corporeal expression — is a denial of thinking in movement" (p. 423).

In challenging this separation, Sheets-Johnstone offers two examples within the experience of the dance improvisation: (1) thoughts of movement (limb extension, a specific quality for the movement, ...) might appear as images or inclinations for the dancer while she is dancing, or (2) she might also integrate movements and gestures that are conducible to everyday-life situations. First, thoughts *of* movement, Sheets-Johnstone claims, emerge in the kinetic flow of the dance without stopping her, overlapping partially to the ongoing process of thinking in movement. They are experienced as discrete events: "they are spin-offs of thinking in movement rather than the result of an ongoing process of thinking in images while moving or the result of any deliberative thinking" (*ibid*.). Secondly, gestures from everyday life might also be incorporated in the dance. From the outside, these gestures might be read as standing for something else within certain cultural standards, however "for the dancer creating the dance, it is the dynamic patterning of movement, its subtleties and explosions, its range and rhythm, its power and intricacy that are foundational, not its referential value as such" (Sheets-Johnstone, 2011, p. 424).

These two examples point out a basic characteristic of thinking in movement, that its *meaning* might be described in terms of a "*kinetic bodily logos*" while undermining the idea that movement arises from externally imposed schemas, or more generally, that mind 'gets all the work done' *before* the body can actually move. Additionally, the author is trying to challenge conceptions for which the dancer movement is (1) a medium through which her thoughts come to emerge and/or (2) a kinetic system of counters for mediating her thoughts.

By the same token, saying that thinking in movement is a way of being in the world and "a natural mode of being a body" (p. 428), the author is also challenging a representational vision of the body, "a body that mediates its way about the world by means of language" (p. 425). I believe that this standing point has at least two important consequences:

- first, Sheets-Johnstone is proposing that we must rethink what it means 'to have meaning'; when we think of thinking in movement (pun intended), we might conceive meaning as something inseparable from what is in the process of thinking or moving, to which one cannot impose a chronological order. There is no thought from which a meaning arises or vice versa, but there is the confluency or composition of the two.

- Secondly, movement is meaningful in itself: when the dancer dances, the only purpose of her movement is the movement itself, the accomplishment of the allusion to the feeling the movement evocates.

Especially in relation to the last point, is of interest Merleau-Ponty's remark about Paul Cézanne's description of himself as 'thinking in painting', which concerns how perception is "interlaced with movement": "it is not a question of *vision* becoming gesture, but of *movement* becoming movement" (Sheets-Johnstone, 2011, p. 429, *emphasis in the original*). Sheets-Johnstone accordingly discusses the distinctions between being the dancer doing the improvisational dance and being the choreographer who creates the dance in terms of artistic process/product and of inside/outside perspective. The thinking of the choreographer in creating the dance is similar to Cezanne's thinking in painting: "Thinking in movement in this choreographic way, she is not only turning "vision into gesture," she is transforming dance into movement — her "vision into gesture" — and movement into dance — "gesture into vision" (pp. 429–430).

In my understanding of it, such an approach allows the author to grasp the experience of thinking in movement in terms of its *global kinetic qualitative nature*, within which no possible division can exist between expression and representation in the dance, between the dance and thoughts of/in the dance, between what I am doing and what I am perceiving I am doing, between before and after, between movement and thought. The expression 'thinking in movement' can be in some sense *read* from left to right but also in the opposite direction. In both ways, it does not just imply a temporal overlapping among the two processes, but the mutual constitution and implications among them. We can capture this saying: *Movement is thinking, and thinking is moving*.

Sheets-Johnstone also grounds observations about thinking in movement from the standpoint of (1) human development and (2) evolutionary heritage, as it is discussed in the following subsections.

3.3.1 Thinking in movement as our primary way of making sense of the world

In her analysis of the primacy of movement, Sheets-Johnstone refers to Lois Bloom's and Daniel Sterne's work in the field of experimental psychological research. Even if their concern is more on relationships between cognition, language and affect, they both recognize (among other scholars) that, in the constitution of a "theory of objects", *movement and change* are crucial for the infant to make sense of the world. This in turn means, in Sheets-Johnstone (2011)'s interpretation, that movement is the foundation of our epistemological construction of the world, in other words, "thinking in movement is our primary way of making sense of the world" (p. 432). By thinking in movement infants get acquainted somehow to objects, motion, space, causality and time.

Bloom's idea of "relational concepts", Stern's "consequential relationships" and Husserl's "if/then relationships" are all "not-language dependent".

> "Moreover they are not simply stepping stones integral to language development, thus essentially "pre-verbal" or "pre-linguistic" phenomena. On the contrary, they are the fundamental backbone of an infant's — and an adult's — knowledge of its surrounding world." (p. 433)

Such an account points out once more the centrality of movement in our understanding of the world. Beyond mentioning the relevance of such studies in the interpretation that Sheets-Johnstone gives of movement, I now underline the main additional points that are of interest in the discourse, concerning the relationship with words and language. The author especially points out:

- that "the *actual dynamic kinetic event* is not reducible to a word or even to a series of words [...] [and] we all have nonlinguistic concepts of [its] dynamics" (p. 434, *my emphasis*);
- that "Thinking in movement is *different not in degree but in kind* from thinking in words" (p. 436, *my emphasis*);
- lastly, that it is not that movement is pre-linguistic, rather *language is post-kinetic* (p. 438, *emphasis in the original*).

Therefore, the primacy of movement over language can be grasped saying that, rather than considering language as *pre-linguistic*, we should speak of the advent of language as *post-kinetic*. This brings us to the point that we have anticipated in §3.1.2, namely the fact that *movement is our mother tongue*.

3.3.2 On behaviour and evolution

From the point of view of the evolutionary heritage, the author takes examples from 'lower animals' such as killdeers and weaving birds in order to disrupt classifications of species (higher/lower) and to continue the discussion about thinking in movement, showing that it is the very quintessence of adaptation and selection:

> "a kinetic bodily logos is at the heart of thinking in movement. It is what makes such thinking spontaneous and contextually appropriate to the situation at hand. It is what ties thinking not to *behavior* but to *movement*, that is, to kinetic meanings, to a *spatio-temporal-energic semantics*. Instinctive behaviors are malleable precisely because they are fundamentally kinetically dynamic patterns and not chunks of behaviorally labeled "doings"." (Sheets-Johnstone, 2011, p. 442)

This last sentence means that the interpretation of behavioural and evolutionary mechanisms under the paradigm of thinking in movement necessarily brings us to the point that the life of humans and animals is to be conceived in dynamic terms (speed, postural orientation, force, direction, etc.) rather than through comportamental wholes (like eating, mating, aggressing, threatening, and so on). In so doing, we can appreciate that behavioural and evolutionary variations exist because *kinetically dynamic possibilities* exist. Thanks to such dynamic possibilities, creatures are distinguished from one another:

> "one creature runs faster than another, is more agile over a rough terrain than another, is more awkward in climbing than another, is less easily aroused or startled than another, is quicker to withdraw than another, and so on. From this essentially kinetic vantage point, the malleability of what are called instinctive behaviors, indeed, their *evolution*, is a matter of movement." (p. 442, *emphasis in the original*)

This directly relates to the discussion around the concept of animation, which we introduced in the first section, because the idea of thinking itself is grounded on it: "Animation is a primary fact of life — and thinking itself, as noted earlier, is itself a form of animation: moving forward, backward, quickly, slowly, narrowly, broadly, lightly, ponderously, it itself is kinetic" (p. 447).

In the end, Sheets-Johnstone's work helps us reconsider and ground the importance of movement and the qualitative nature of self-movement that can elucidate the interconnection between movement and thinking. Ultimately, movement gives us the possibility to bridge the different scales of intrapersonal (self-perception and movement), interpersonal (couple of individual, or small group) and evolutionary development.

In particular the present chapter has brought forth the following points:

- we should not consider movement and thinking as separated but see that the two processes mesh into each other and fuel each other.

- Movement is *meaningful per se* (just as we are normally driven to consider thinking as relevant *per se*, especially in the context of mathematics).
- The idea of thinking in movement can be the ground for a unified vision about movement and thinking.
- Focus on qualities of motion can disclose the analytical structures of movement.

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[1] In Husserlian terminology, the descriptive analysis of experience brings forth that the ability to move oneself encompasses the range of "I cans" that are foundational for any "I do". Therefore, in Husserl phenomenology, the consciousness of this bodily motility grounds one's sense of agency in the world, namely the feeling of being agent in the world of things. Nevertheless, according to the Internet Encyclopaedia of Philosophy, "motility is a broader concept than agency in the strict sense whereby an "agent" would be actively, explicitly involved in initiating and directing the action throughout" (https://www.iep.utm.edu/husspemb/#SH4c; accessed on November 9th, 2018). A wider interpretation can be thus given to this term, which might refer to an unfolding of movement that does not entirely depend on the will of the human body. The work of Sheets-Johnstone deeply relies on the approach proposed by Husserl, which is indeed one of the main sources of her work. The chapter will in fact present some issues that are familiar to readers who know the theoretical account of the latter author, but it will more directly focus on the consequent interpretation of the former author.

[2] Chapter 11 of "The Primacy of Movement" (Sheets-Johnstone, 2011), which is eloquently titled "What is it like to be a brain?", discusses such issue in detail. The title paraphrases Thomas Nagel's article "What It Is Like To Be a Bat?" (Nagel, 1974), in which the author attempts to specify the so-called subjective experience, or what is for a bat to be a bat. He insists that the problem of consciousness does rely on the mind-body problem and proposes to devise an objective phenomenology.

[3] A flatlander is an inhabitant of Flatland, the world populated by geometrical figures described in Abbott (1884).

[4] This is only one of the reasons why a comprehensive account of movement is hard to reach. As we read in Moore and Yamamoto (2012):

> "The inherent complexity of movement has contributed to the difficulty of capturing all the relevant details. Every human movement involves activation and coordination of different parts of the body, displacement through space, and use of dynamic energy. Multiple changes in all these elements occur simultaneously as the movement progresses through time and space. At each instant, these configurations are appearing and rapidly disappearing. Recording all these changes proved to be a daunting task." (p. 7)

This and other methodological issues that concern the capture and description of movement will be discussed in depth in Chapter 5.

[5] The reader is invited to perform the movements to appreciate kinaesthetically the proposed example. The exercise presented by Sheets-Johnstone (2011) is the following:

"In particular, we ply our trade now in order to elucidate cardinal structures of kinesthetic consciousness. We do this by taking a very simple movement, a movement that is basically familiar — an overhead arm stretch — but slow it down and further heighten our sense of movement by making a formal beginning: we start by closing our eyes, by dropping our head so that our chin falls toward our chest, and by resting our hands in our lap. From this beginning position, we lift our arms from the elbow so that our upper arms move upward and our hands come off our lap. We continue that upward movement without a break by extending our forearms upward and overhead, and finally by extending our fingers upward and overhead. At the same time we do all this, we slowly raise our head from its dropped position to the point that our chin faces upward toward the ceiling. We then reverse the movement, first by letting our elbows flex and our chin begin moving downward, and then by simply continuing the movement of arms and head downward until we come to our original position. We do this sequence of movements three or four times slowly, by ourselves, keeping our eyes closed and sensing the phenomenon of self-movement." (pp. 121-122)

[6] Examples from mathematics education literature are studies about mathematics-related affect and the emotional dimension of mathematics, which tend to assign particular emotions/feelings to particular students, or to speak of specific relationships between beliefs, attitudes and feelings in representational ways (e.g., Hannula, 2012; Radford, 2015; Zan, Brown, Evans, & Hannula, 2006). For a deeper argument, attentive to affective ecologies that go beyond specific focus on the individual, see de Freitas et al. (2018).

[7] *Oxford English Dictionary*.

Intermezzo: Virtual and movement

This intermezzo intends to capture the unfolding of the theoretical highlights, which have been presented until this point in the dissertation. In Chapters 2 and 3, we delved into virtuality and movement respectively. These concepts have grounded the theoretical framework of this study together with the inclusive materialist perspective (see Intermezzo: Inclusive Materialism), which offered an initial background for this research, and the vision of concepts detailed inside Chapter 1, with specific focus on the concept of function. The previous Intermezzo has introduced the idea of movement in mathematics education and a broader interest in the concept of movement.

Virtuality has mainly been investigated following DeLanda's reading of Deleuzian realism and Châtelet's vision of mathematics and mathematical activity. It opened the ground for discussing how concepts are dynamic arrangements subjected to formation and deformation in non-deterministic ways, as underscored through the idea of multiplicity. Drawing on the work of Châtelet, I have also highlighted some features of the virtual, namely its obscurity, allusivity and elusiveness, its mobility. By reviewing how the concept of virtuality has been exploited in mathematics education, we further discussed how the vision of Châtelet might be pedagogically relevant to pursue. For Châtelet, in mathematical activity the virtual is revealed through the interplay of the gestural and the diagrammatic, that is, through the body. This aspect, which bridges the creative production with the indeterminate dimension of the body, highly resonates with the "realm of possibilities" of perceptuo-motor engagement (Nemirovsky et al., 2012) and the relational nature of human perception (de Freitas, 2016), that is, with the potentiality of the human body.

The concept of movement has been deepened starting from the work of Sheets-Johnstone on the primacy of movement. Her work helped us engage with the issue of movement in a broad perspective, even touching on fields that are very far from mathematical content and context. Following Sheets-Johnstone, we dwelled upon some ideas that can be reversed (like in the case of the body-mind dualism) if we take movement as a complex and enigmatic but central concept to understand life. Understanding movement is crucial from a developmental perspective but comes to get significant especially in relation to thinking. In particular, we explored the subtle process of "thinking in movement" as that in which movement informs and sustains thinking and vice versa. This vision sheds light on the profoundly dynamic nature of thinking. Whether we turn to mathematics and mathematical activity, and the activity of thinking mathematically, we can investigate the mobile and kinaesthetic nature of mathematical thinking through the body and the virtual.

Movement and virtuality are both elusive, enigmatic, problematic dimensions of life and mathematics, each of which goes beyond our capability of capturing or possessing a comprehensive view of them.

In particular, thinking of movement through the concept of virtuality, that is, thinking about mathematics as first and foremost mobile, generates questions about how movement enters the realm of mathematics and also the way that moving, or being bodies in movement, partakes in the doing of mathematics. To summarise, I argue that making sense of how we think in movement does inform the way in which we might learn or do mathematics. This also implicates the title of the dissertation: the idea of mathematical thinking in movement is one that brings forth my attempt to capture the manifold potentiality of thinking with main attention to the bond of movement and the virtual dimension of mathematical concepts. Movement and virtual sustain each other beyond tangling with each other. Therefore, mathematical thinking in movement captures a duality of grasp within the same expression: mathematical *thinking in movement* (that is, thinking in movement in the doing of mathematics) and *mathematical thinking* in movement (that is, the process of thinking mathematically as emerging from movement).

The previous chapters provided the reader with an entanglement of different theoretical standpoints: from realist philosophies (DeLanda on Deleuze, Châtelet, Barad) to phenomenology (Sheets-Johnstone) through the post-humanist sensitivity of inclusive materialism (de Freitas and Sinclair). It is the entanglement that informs the theoretical framework of my research and creates potential lines of flight for my commitment to movement. In fact, the concept of movement traverses all these theoretical approaches, beyond their difference or distance: their focus on it uncloses a common ground on which they can be nourishing each other. Rather than thinking of them as incompatible, we can put them to work together on such a shared ground that makes space for the epistemological-ontological vision that I pursue in this research.

4

Design

In this chapter I will introduce the reader to some theoretical and empirical aspects underlying the design of the classroom activities, which are at the core of the research project. The interventions were planned and implemented in grades 4, 7 and 10 and involved software for graphing motion, WiiGraph. Prior to the interventions, we held a pilot study in a class of grade 4 learners.

In particular, I will delve into the theory of design and teaching experiments in mathematics education to better position the interventions. My line of work will be specifically situated into recent theory that considers classroom experiences as *apparatuses* (de Freitas, 2017), and thus aligned with the theoretical commitments exposed in Chapter 1.

Then, the chapter proposes the main principles that characterised the design of the activities, considering differences and similarities about the choices made for different grades. These aspects will be framed by considerations on task design as they are currently studied in mathematics education research.

At the end of the chapter, an overview of the longitudinal research study will be presented. I will touch on this by emphasizing the role of the experiences with the software during the course of the teaching experiments, proposing a diagram that aims to capture their structure in terms of time, concepts, experiments, and types of interactions.

4.1 Meta-considerations

As researchers in mathematics education, meta-reflection on our practice in the classrooms leads to observations about how our positioning unfolds in different directions, from the design of tasks and educational materials, the planning of interventions, the setting of learning environments, the implementation of materials in the created setting, to the analysis of the previous phases, with focus on teaching and learning processes. Schoenfeld (2008) observes that, in the whole process, "whether or not researchers believe that they have theoretical perspectives and biases, they do" (p. 479). In addition, since the very beginning of my engagement with the research process, as a newcomer in the field, I experienced how the process is influenced by theoretical assumptions, is connected to material and practical circumstances and might affect from a single class of students to the research community (e.g., when the process is elaborated and shared).

As Sinclair (2014) pointed out in threading some influential ideas and motivations within research on the use of new technologies in mathematics education over the past four decades, researchers' interest towards the theorization of learning mathematics with digital technology has been increasing. Schemas that guide researchers in the process of interpreting the tool-student relationship have been elaborated. For example, when adopting an instrumental or documentational approach (Gueudet & Trouche, 2009; Guin & Trouche, 1999) the development of *a priori* and *a posteriori* analyses of teachers and students' interactions in technological rich environments are crucial steps (a question of methodology). Another well-spread approach, Design-Based Research (Bakker & van Eerde, 2015), aims at developing theories about learning and the means that are designed to support learning, by implementing several cycles of three phases each (preparation and design, teaching experiment, and retrospective analysis; a matter of method). Another example is given by the TPACK framework, which wants to incorporate pedagogical and content knowledge into the technological knowledge of teachers (Koehler, Mishra, & Cain, 2013) to understand teacher knowledge in the context of technology use and implementation.

In spite of differences in orienting the framework (theory) that constitute the ground for such approaches, one main common point we can recognize, which they all try (even implicitly) to address, is the complexity of the process of teaching and learning (mathematics) with technology and the mutual relationships between practice (methods), theory, methodology, and mathematics (epistemology) (Stinson & Bullock, 2015). There still are significant differences in the ways in which this is pursued, which exactly rely on the theoretical assumptions on which they are grounded.

This is quite apparent in Sinclair (2014), as she engages in a diffractive reading on digital technologies, mathematics and the body to make the inextricable bond between these nodes emerge and to highlight the need for a shift from epistemological to ontological commitments. As introduced in the first Intermezzo (Inclusive materialism), the new materialist perspective elaborated by de Freitas and Sinclair (2014) addresses this matter, while challenging well-established assumptions in mathematics education that involve the role of the body and the representational nature of symbols, diagrams and language. We have already noticed, that, in this sense, the idea of learning assemblage helps us reconceive the nature of the human body while drawing attention to how the boundaries between bodies are mobile and always shifting. The moving assemblage of students, paper, pencil, technology is sustained by forces that shape and mutually constitute the evolution or becoming of the assemblage. The authors' approach may sound non-operational to those who are more familiar with mathematics education paradigms that look for evidence and categories in speaking of understanding. Nevertheless, this approach can be put in motion when trying to speak differently about "how bodies are assembled through activity", which is far from studying "the learner who drags the triangle on the screen as an enclosed body that knows, acts or feels independently from the mouse, the screen or the digital interface more generally" (de Freitas & Sinclair, 2014, p. 15). The perspective allows seeing the body of the discipline in a completely different manner, which is able to capture the continuous becoming of the mathematics through the activity. All the approaches we quoted above lead to a conception of knowledge as something fixed or immobile, across which the learner moves independently, even in the attempt of considering dialectical relationships among different aspects of knowledge (as it is within TPACK, for the pedagogical and technological sets of resources: e.g., Koehler et al., 2013).

Prompted by these theoretical and meta-theoretical elements, particularly by the vision of concepts as devices that actualize the virtual (as exposed in Chapter 1), I propose some ideas for the design of learning activities, which I articulate in the following. Turning attention to mathematics more as a multiplicity of articulations along the lines of virtuality than as a simple collection of conditions and cases necessarily shifts the focus of design to that which is significant for the students to encounter in the classroom interventions. The task design of this study exactly aims at promoting this vision and is thought of as engendering some specific mathematical content as well as ways of (methodologically) favouring those encounters, according to my theoretical commitments (Chapters 1 to 3). In light of the entanglement of the different elements that constitute the body of research (highlighted throughout this section), I articulate the main design principles of my research along different lines: first, touching on theoretical concerns about teaching experiments, then recalling how the epistemological sphere intervenes in our design and, finally, focussing on the specific study design. The next section deepens these concerns, taking into account entanglements with theory, methodology and data.

4.2 Theory of didactic interventions

Interventions in classroom context are named classroom-based interventions (Stylianides & Stylianides, 2013) when they are held with a twofold aim that bridges the pragmatic and theoretical objectives of research studies. On the one side, a core aim of classroom-based interventions is the improvement of classroom practices on one or more specific topics; on the other, the dual aim is the contribution to research knowledge on specific classroom phenomena, which are observed in real contexts (i.e., inside actual classrooms), while such knowledge is built upon theoretical investigations of those phenomena. In our study, we aimed at improving classroom practice about the introduction of the concept of function via a graphical approach. This meant, for us, to structure the classroom activities, through which the students could encounter functions and graphs, in an unconventional way with respect to standard classroom practices. Even though in some classroom-based interventions there is a tendency to diagnose specific behaviours in the classroom and to propose solutions to them, this was not a focus of our study. Design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gorard, Roberts, & Taylor, 2004; Kelly, 2004) share with classroom-based interventions the pragmatic and theoretical purpose of studying both the design and the resulting ecology of learning, which might be very diverse according to the settings and the groups of researchers, teachers and classes involved. Design experiments are devised with a slightly more emphasized interest on the innovation they can bring to the context (new forms of learning, new methodology or environment). A distinctive feature of design experiments is in fact a highly interventionist method. Following Steffe and Thompson (2000), the teaching experiment is

> "a conceptual tool that researchers use in the organization of their activities. It is primarily an exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics. […] It is a dynamic way of operating, serving a functional role in the lives of researchers as they strive to organize their activity to achieve their purposes and goals. In this, it is a living methodology designed initially for the exploration and explanation of students' mathematical activity." (p. 273)

Such experiments also serve the researchers to build accounts of how students learn specific mathematical concepts, provided that mathematical activity in school occurs as a result of students' participation in teaching. Both design and teaching experiments are part of design-based research methodology (Bakker & van Eerde, 2015; Barab & Squire, 2004; Kelly, 2004), according to which the design of educational materials and learning environments is interwoven with testing or developing theories about learning. In particular, design-based research studies have a cyclic structure, so that each cycle involves designing instruction, implementing that design in the classroom, and analysing and rethinking the new design in light of the previous session. This allows for consecutive adjustments, in order to achieve ongoing improvements of the initial design. In teaching or design experiments, "[t]he overall goal in enactive successive design and analysis cycle is to test and improve the envisioned learning trajectory formulated during the preparation phase" (Cobb, Jackson & Dunlap, 2017, p. 213).

Moreover, in a teaching experiment, the role of the teacher-researcher is crucial, even if the focus of attention is on students' reasoning (Steffe & Thompson, 2000):

> "In a teaching episode, the students' language and actions are a source of perturbation for the teacher-researcher. It is the job of the teacher-researcher to continually postulate possible meanings that lie behind students' language and actions. It is in this way that students guide the teacher-researcher. The teacher-researcher may have a set of hypotheses to test before a teaching episode and a sequence of situations planned to test the hypotheses. But, because of students' unanticipated ways and means of operating as well as their unexpected mistakes, the teacher-researcher may be forced to abandon these hypotheses while interacting with the students and to create new hypotheses and situations on-the-spot." (p. 280)

Therefore, the classroom interactions are thought of as malleable and the students' actions might influence the hypotheses carried out at the beginning of the study by the researchers. More often than not, these interactions are thought of as fostering learning processes, and "the interest is in understanding the students' assimilating schemes and how these schemes might change as a result of their mathematical activity" (p. 288). The researchers in turn use such schemes to construct models that explain students' mathematical activity.

De Freitas (2016) is critical towards design and teaching experiments when she says that "[a]ll too often, design experiments pay tribute to a positivist image of empiricism in which naïve objectivity is assumed, and knowledge is treated as representation rather than activity" (p. 167). She proposes to rethink the nature of didactic interventions in terms of Karen Barad's (1996, 2003, 2007, 2011) idea of the *apparatus*. An apparatus is a more or less complex instrumental device that is used in experimental physics (and might also be involved in thought experiments). According to Barad, an apparatus is not only a way of verifying or testing hypotheses, but can engender the new, therefore it operates at both material and ontological level. De Freitas argues that design experiments *are* themselves particular apparatuses that (1) re-assemble the concepts in new ways and (2) work at both material and conceptual level in learning assemblages. She mainly draws on Barad's *agential realism*, which embraces mutual relationships between epistemological and ontological commitments. Barad explains that "*Realism is not about representations of an independent reality, but about the real consequences, interventions, creative possibilities, and responsibilities of intra-acting within the world*" (Barad, 1996, p. 188, *emphasis in the original*). The term *intra-action* is a "*re-working of the traditional notion of causality*" (Barad, 2003, p.815, *emphasis in the original*), and is used to query the assumption that matter and meaning need *a priori* distinction, and to challenge classic visions that postulate objects primary to relations. This is not the case if we take some examples from physics, like the double-slit experiment. This experiment made apparent to scientists the wave-particle duality of light, i.e. that light manifests particle behaviour under certain circumstances (and wave behaviour, by means of interference patterns, under different circumstances). Examples like this shed light not so much on the way that the measure depends on the instrument we are using to perform a measurement, but rather on the ontology of the process of measurement itself. In Barad's words, "it is not so much the case that things behave differently when measured differently; rather, the point is that there are only phenomena – the intra-action of "apparatus" and "object" in their inseparability" (Barad, 2011, p. 143). Hence, intra-action is used instead of interaction to emphasize the shift to this vision. Drawing on observations by the physician Niels Bohr on such arguments, Barad posits the need for a different concern about the indeterminate nature belonging to matter in the first place (*new materialism*), specifically about the non-deterministic relationship between matter and meaning, which necessarily leaves out discourses about objectified knowledge. In turn, such vision implicates a different ethics of knowing, so that the researcher positioning is not about observing or reflecting, rather her entanglement with methodology and data entails a more sophisticated and subtle discourse and the need for different methodological approaches. In particular, Barad proposes a *diffractive methodology*, which entails seeing new patterns by reading important insights through very diverse sources (particle or wave nature). As it is discussed for the double-slit experiment, which reveals patterns of interference and characteristic wave behaviour for a light ray, say van der Tuin (2011), a diffractive method entails a "reworking" of the concepts that "structure these insights or appear in the traditions of thought from which they stem" (p. 27).

Barad (2003) also discusses how, for Bohr, theoretical concepts (like that of *position*) are not ideal (abstract) but rather are "*specific physical arrangements*".

> "For example, the notion of "position" cannot be presumed to be a well-defined abstract concept, nor can it be presumed to be an inherent attribute of independently existing objects. Rather, "position" only has meaning when a rigid apparatus with fixed parts is used (e.g., a ruler is nailed to a fixed table in the laboratory, thereby establishing a fixed frame of reference for specifying "position"). And furthermore, any measurement of "position" using this apparatus cannot be attributed to some abstract independently existing "object" but rather is a property of the *phenomenon*—the inseparability of "observed object" and "agencies of observation."" (p. 815, *emphasis in the original*)

Summing up, in the agential realism proposed by Barad, *phenomena* carry an ontological primary status as they are ontologically primitive relations.

De Freitas (2016a) follows Barad to argue that (in mathematics education) a design experiment may operate *as a diffractive apparatus*, that is, as an instrumental device that makes interference behaviour (diffraction) emerge out of specific practices (exactly like in the case of the double-slit experiment). In de Freitas' words: "Insofar an experiment involves a diffractive device, the experiment becomes a means of mutating concepts and reassembling the world. Such an experiment has consequential meaning and cannot be described as simply means to test hypotheses" (p. 157). The author shows how a design experiment with children using a technological device also involved an ontological work of the students and the technology upon some concepts, engendering the new. In fact, "mathematical concepts are not determined or preformed prior to the experiment" (p. 171): as "physical arrangements, then they are themselves involved in coordinated motion" (p. 168). De Freitas argues that the teaching experiment she takes into account helps us understand how new forms of relationality, causality and time are being produced through the particular apparatus (students working with a graphing motion device).

Therefore, if we rethink our didactic interventions in these terms, we are also asked to rethink how the mobility and the material dimension of concepts are addressed in such experiments. In light of a methodology that takes into account this diffractive vision, the positioning of the researcher is also relevant, as the very notion of observation is put in motion and always constitutes an intervention at some degree. In §4.4.3 I will discuss the consequences that this vision has for the methodology of my research, in particular regarding task design. As a main point, de Freitas (2016a) observes that designing an apparatus that engages students in mathematical learning actually "recruits the student's bodies as part of the apparatus" (p.166). Drawing on this vision, we now turn to the implications in terms of the epistemological/ontological concerns already introduced in Chapters 1 and 2. The indeterminacy offered by a diffractive methodology in fact opens up the scene for studying how the concept is a vibrant apparatus that is put into motion inside the mathematics classroom.

4.3 Epistemological concerns

To start with, in Chapter 1 we have used de Freitas and Sinclair's (2017) perspective on concepts as generative devices to propose a shift about the ontology of mathematical concepts, and we proposed a discussion on the concept of function, based on contingent examples that directly relate to a vision of mathematics as practice. Hence, in Chapter 2 we have taken the metaphor of the manifold to characterise the virtuality of mathematical concepts, which troubles the image of mathematics as abstract and immobile. The virtual constitutes an ontologically relevant bridge between the mathematical and the physical and grants concepts with an extreme mobility. Moreover, the concept of the virtual prompts us to reconsider mathematics in terms of latent virtualities hidden in any mathematical encounter. Exactly this aspect made new question to emerge for me: What if, instead of constructing and de-constructing concepts, or considering a consequential and built-in order existing within concepts and the resulting developmental processes of students, we try to address the mobility of the concept? How do we make room for the encounter with the virtual to take place? What does this might mean for teachers and researcher in practical terms of task design? This kind of questions was crucial within the more pragmatic phases of our task design, as we will detail in the next section.

As a starting point, we draw on De Freitas and Sinclair (2014), who propose that re-examining familiar mathematical ideas embracing virtuality might be achieved through rethinking the concepts in terms of mobility and vibrancy. Of course, we are also in debt with the Italian tradition in mathematics education research, which has been prolific in the context of dynamic explorations of concepts. To name just two of them, Emma Castelnuovo and Federigo Enriques had proposed dynamical approaches to mathematics teaching, which emphasize the role of movement as a starting point for recognizing relationships and working with mathematical structures since the early ages. Lastly, we position in the Italian trend of the 'research for innovation' (Arzarello & Bartolini Bussi, 1998), and we share with it a distance from the positivistic view of the natural sciences that has characterised research in education, especially in the past.

In this study, we addressed the idea of mobility by involving a specific technology, WiiGraph, which makes functions *move* in the Cartesian plane demanding students' movements in space to create graphical representations. Building on the conditions created by the technology in the mathematics classroom, we also centred task design and the structure of the didactic interventions on relations or transformations between graphs. This allowed us to relate functions rooted in movement and graphs by means of movement, as I will detail in the next section. Complementary to this point, rethinking mathematical concepts in terms of their virtuality collides with a classical, structural vision of learning; this aspect requires us to give account for a critical overview of task design in mathematics education, which is carried out in the next section.

4.4 Task design

Task design has been the focus of recent research in mathematics education (Margolinas, 2013; Watson & Ohtani, 2012). Drawing on a consistent literature review in the field, Margolinas reports that (1) it is necessary to have theories about learners' intellectual engagement to have successful design; and (2) most design principles include the use of several representations, several kinds of sensory engagement, and several question types. In fact, concerning the first point, the design of activities is part of the relationship between research and practice, whose nature and strength has become a more prominent matter of attention and concern in the last years, as outlined by Silver and Lunsford (2017). Concerning the second point addressed by Margolinas, it brings forth the increasing interest in task design in educational contexts in which some digital technology is used or in the design of digital learning environments.

The design of a sequence of tasks (with or without technology involved) is usually structured around concept de-construction and ideas of learning trajectories and progressions in mathematics, which have been emerging through a variety of approaches (Lobato & Walters, 2017; Simon, 1995). Whether attention is drawn to cognitive landmarks, ways of communicating, schemes and operations over time, ways of reasoning toward a particular learning goal, types of collective practice, curricular coherence, or strategies and performances (lines of research), these ideas are discussed as prominent ways of capturing unifying levels or developments of mathematical thinking and understanding (J. Confrey, Maloney, Nguyen, & Rupp, 2014). Discourse offers common views of unidimensional, vectored paths characterised by increasing sophistication, ordering, progress and knowledge growth, lower to higher, informal to complex degree.

This universal character does not seem to do justice to the contingent material and embodied dimensions of teaching and learning mathematics in the classroom, though. In fact, critiques regarding learning trajectories and progressions, despite their potential, concern how research is framed theoretically but also those who participate in the research and the types of tasks that are employed (Lobato & Walters, 2017). Interestingly, sequences of conceptual attainments are at odds with the manner in which progress is characterised in scientific disciplines, for example they forget that the formation of wrong ideas has often been generative for later progress (Sikorski & Hammer, 2010). Also, when misconceptions are included, they are so only in lower levels and without reference to how they might be a (re)source for later development. Problematising these aspects is crucial to pursue a different line of research, which is not aimed at defining a learning trajectory but wants to explore a different approach to prime mathematical encounters in the classroom.

To do this, we follow again Sinclair and de Freitas (2014), who criticise the idea of learning trajectory as one that demotes the creative force of the activity and the mathematics. However, the inclusive materialist framework is not a framework *for action* or ready-made for use: it does not propose a structure to design tasks according to their theoretical underpinnings. Nevertheless, the examples they propose (e.g., the ones discussed in Chapters 1 and 2) gives us a direction for investigating how this might be done. We may also notice that such examples involve some kind of diagram or device and an onto-epistemological work on concepts.

For this reason, in the following, I illustrate the main principles of our design, which partly rely on an interpretation of the inclusive materialism framework and the articulation of activities in the teaching experiments afterwards.

In terms of task production, it seems significant to notice that what we referred to as "the mobility of concepts" arises from variations and modifications occurring to the concept itself, as it is experienced and encountered by students. In Châtelet's perspective, as we detailed in Chapter 2, it is the virtuality of physico-mathematical concepts that grants them a profound mobility; the mathematician excavates new meanings out of the manifolds of potentiality that is entangled with her gestures and diagrams. Reformulating this slightly, as regards our interest in a graphical approach to functions, a relevant issue is that of thinking about different graphical shapes as they arise from plane transformations. To say it differently, we might think of different graphs as they emerge from the stretching, deforming and moving on a surface of another graph.

This is relevant also because we can always think of couples of graphs as one emerging from the other via a transformation. Each transformation in the Cartesian plane is itself a function, which is applied to particular points on a plane, but one that creates some relationships between things that are already expressing some relationship (two graphs). Somehow, then, comparing, contrasting and transforming graphs in such a way is already thinking of the graphs (as a couple) in terms of what I would call a *relational relationship*.

This discourse might resemble the principles proposed by variation theory (Marton, Tsui, Chik, Ko, & Lo, 2004) for what concerns its pragmatic aspects but, as we will briefly explain, it has very different implications. Very briefly, Marton and colleagues observe that we discern the main features of an object by noticing and experiencing how it is varying. They argue that "[b]y experiencing variation, people discern certain aspects of their environment; we could perhaps say that they become "sensitized" to those aspects" (p. 11). The authors assume that "human beings cannot discern a feature without experiencing variation in a corresponding dimension" (*ibid.*). According to variation theory, then, learning occurs whether one becomes aware of new or critical features that are crucial to define an object, when one notices the structures that lie behind a problematic situation and starts making conjectures about them.

For example: to identify the critical features of a linear graph, one should experience how it may vary in different contexts, noticing different intercepts, the changing slope, the line-up of points which is preserved, and so on. In my understanding of it, a starting point for designing tasks according to variation theory might be to consider and list all the possible cases along with these variations occur (within the four categories Marton and colleagues propose).

What instead drives us for the designing of the activities is the *potentiality* of the line to be stretched, modified and problematise or, in a word, the potentiality to be put into motion by means of experiments and explorations. To experience change therefore means to encounter and make sense of how and why it does vary the way it does, not constraining change to cases, but creating space and openness for variations and modifications to occur. In doing so, emphasis is put much more on the relations and how they are constituted and changing, rather than on the creation of a path to be followed thanks to the design. This is of course very general, so I will provide the reader with specific examples. What matters the most is that, in this context, experiments are crucial points of the design: they are, at the same time, creating indeterminacy and fuzziness and are generative elements for classroom activities. As Nemirovsky and colleagues (1998) notice, no experiment is alike another one, and "[n]o two individuals' graphing processes are identical, no two bring the same resources to a new situation" (pp. 123–124), and this complicates the structuring of a precise schedule for the activities as well as it requires extreme flexibility from the point of view of the teacher/researcher.

To summarise, one main principle in the design of activities was that of creating encounters with the mathematical concept of functional relationships via graphing motion. This meant for us making room for the students to be engaged with meaningful activities, which allowed for encounters with the concept of function through a (mainly) graphical approach by means of bodily interactions. In light of the meta-theoretical aspects, which we traced throughout this chapter, we also wanted to take into consideration the presence of technological tools not just in terms of its influence on the teaching and learning process, but with an eye on capturing the ways in which tools engender an onto-epistemological work and perturb the concepts and the activity.

This resonated with a vision that insists on considering the relevance of perceptual nuances that occur in the student-tool relationship, even though it stems from a different theoretical perspective. In Noble, DiMattia, Nemirovsky, and Barros' (2006) words, "one's own sensitivities and perceptions and one's own physical abilities are shaped through one's growing competence with a tool" (p. 434). According to this line of thought, which is very inspiring for designing tasks with graphing motion devices, the fluent use of a mathematical instrument is related to the adoption of a *tool perspective*, which "involves emulating the tool's sensitivity to certain aspects of motion and not to others, ascertaining conditions under which the tool is useful, and recognizing patterns of significance in the tool's products" (Nemirovsky et al., 1998, p. 125).

This does not mean that the aim of the experiments is that the students developed a specific ability or skill in using the tool, nor are we suggesting that this is crucial for an understanding of the concept of function. Rather, we are interested in the ways in which we can use the software to create space for the concept of function to be encountered by students by means of experiencing spatio-temporal relationships in the context of modelling motion. We stress that, in such experiences, the role of the body and its movement (that is, proprioceptive and kinaesthetic engagement) is a crucial aspect to take into account for designing tasks and this is in line with the main interest of this study. Moreover, we trace a sequence of tasks that aims to develop ideas around the concept of function not for identifying a hypothetical learning trajectory, rather with the theoretical aim of attending to "the ontological work that children and technology are doing when they are learning" (de Freitas, 2016, p. 167). Since the technology in use is one that allows for graphing motion, we now turn to extend what we have already discussed in §1.5.2 about graphing motion technologies as tools for introducing the concept of function inside the mathematics classroom.

4.4.1 Graphing motion

Graphing motion activities have been largely investigated in mathematics education research since the 90s, through the use of motion detectors and other technology (e.g. Nemirovsky et al., 1998; Yerushalmy & Shternberg, 2005; Radford, 2009; see §1.5.2 for a discussion). Researchers have been studying the ways in which the interaction with this kind of tools may stimulate mathematical thinking while taking advantage of perceptuo-motor activity. Even though different researchers have offered different conceptions of function, these studies generally share the vision of covariation as a foundation for function in mathematics (see Thompson & Carlson, 2017). Focus here is on highlighting features of graphing motion activities with a specific technology: a software application named WiiGraph[1]. This technology allows for the creation of different types of graphs while two users move each a controller. Drawing on Nemirovsky and colleagues (2013), WiiGraph is a mathematical instrument, that is, "a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics" (p. 376). Implicating movements of the controllers by two people in an interaction space, activities with WiiGraph also implicate bodily proprioceptive and kinaesthetic experiences both with the devices in use and with the graphical lines and symbolic operations provided by the technology. Nemirovsky et al. (2013) unfold the powerful idea of mathematical instrument to speak about fluent use and mathematical expertise as inseparable from perceptual and motor aspects implied in the activity with the tool. The authors propose that to acquire fluency with a mathematical instrument, one necessarily traverses the gradual integration of perceptuo and motoric aspects of the activity: they advocate "for a perspective on tool fluency that is explicitly informed by an embodied approach to mathematical thinking and learning, a perspective that [...] entails that (a) mathematical thinking is constituted by bodily activity at varying degrees of overt and covert expression, and (b) mathematical learning consists of transformations in learners' lived bodily engagement in mathematical practices" (p. 376).

While these researchers are interested in studying fluency with the instrument in the informal context of a scientific exhibition, we focus on the more formal context of the mathematics classroom. In the design of tasks, the vision of Nemirovsky and colleagues helped us draw attention to the kind of engagement and practices that activity with the technology might favour within the classroom (e.g., strategic thinking, collaborative dynamics, use of material resources, etc.)[2]. We centre on these aspects as a way of discussing challenging lines of flight on covariation, function and families of functions and the issue of designing activities for students from the early years to secondary school. In the next section, we will introduce the technological tool for which we designed task and activities.

4.4.2 WiiGraph

WiiGraph is part of a family of mathematical instruments that engages the students in ample movements to create mathematical representations based on their motion. It is a software that leverages two WiiRemotes to graphically capture their position (their distance from a sensor bar). Despite the many possibilities and environments that it offers, we are interested in considering those options that allow for graphing motion of people who act in an interaction space. When a session starts, a Cartesian plane appears on the computer screen and one or two lines that capture the two remotes' position over time, begin originating in real time (Figures 4.1 and 4.4).

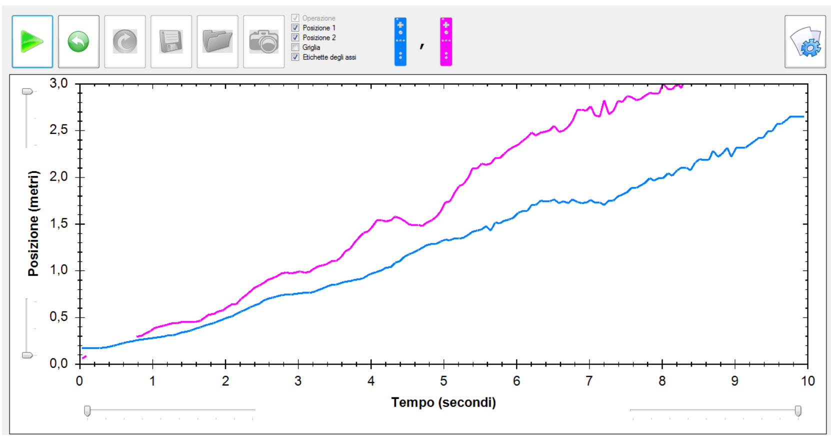


Figure 4.1. Line graphs

I briefly illustrate two main options of WiiGraph: *Line* and *Versus* (cf. Appendix A for a more detailed presentation of the software). *Line* furnishes in real time two position-time graphs that capture the distance of the Wiimotes over time from the sensor (see Figure 4.1). Temporal and spatial ranges can be set and modified for the Cartesian axes. This in turn implies a specific time interval for the motion to be performed, which can be set, for example, to 30 seconds, and space constraints for the two users' movements within the interaction space (e.g., at most 10 feet far from the sensor). Labelled *a* and *b* the two distances, the software displays the lines $a(t)$ and $b(t)$ differently coloured on the screen. The lines appear in real time as the users move in space, whether they correctly point their remote to the sensor bar. That this happens is visually signalled with a dot of the corresponding colour, which appears on the screen. When the dot disappears from the screen, there is no correct interaction between the remote and the sensor bar and the software is not able to capture the remote's position at that time.

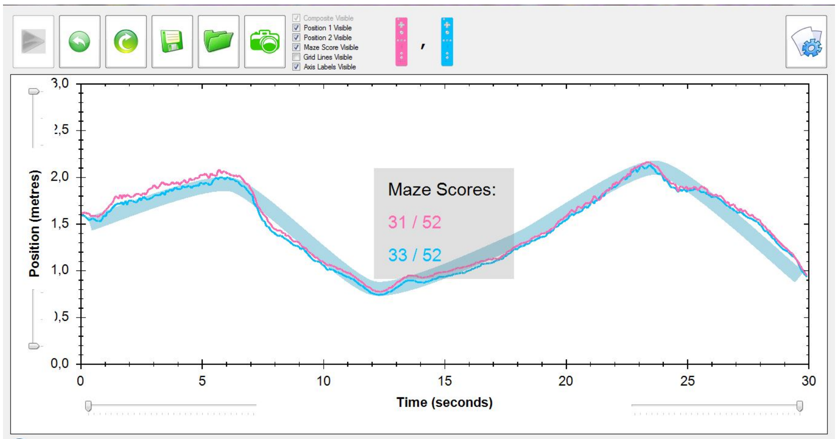


Figure 4.2. Make Your Own Maze! target and Line graphs

Additionally, selecting the *Make your own Maze!* modality allows for the creation of a target maze to be traversed by the graphs associated to the users' movements. The maze can be built choosing a number of inflection points, a certain value for its thickness and tension, therefore a particular graphical arrangement for the maze, which is visible on the screen as a tick light blue line (Figure 4.2). The target is chosen at the beginning of a session and remain fixed on the screen for the whole duration of the session. At the end, each user gets a score based on the rate by which the created graph traverses the maze.

Within the *Operation* modality, a third coloured graph is shown on the screen: in particular, the addition *a*+*b* implies a third graph of position over time that depicts in real time the sum graph $a(t)+b(t)$; in few words, the new graph shows instant by instant the sum of the two distances captured by the two other graphical representations (Figure 4.3). It is also possible to choose among other simple operations (subtraction, multiplication, or division), with an analogous result (a third graph that complies with the chosen mathematical rule).

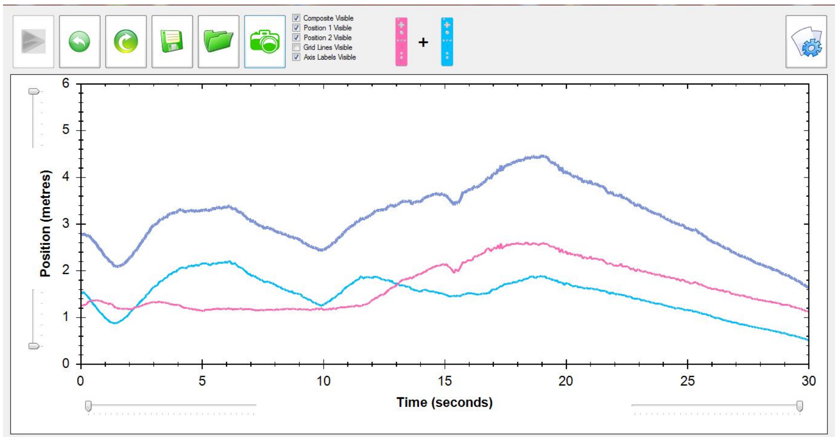


Figure 4.3. Sum graph

The second option we draw attention to is *Versus*, which allows for the creation of a single graph on a Cartesian plane with isometric axes, depending on both users' movement (see Figure 4.4).

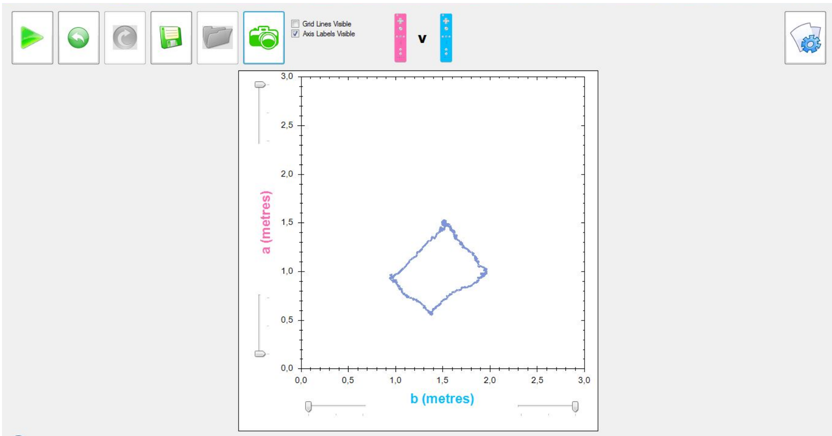


Figure 4.4. Versus graph

Versus graphs are obtained by pairing two functions of position versus time, one for the horizontal axis and the other for the vertical axis (like in the Drawing Machine, already described in Chapter 1, but through the software and not through mechanical articulation of the machine). These functions again capture the positions over time of the two remotes.

The composition of the two movements in such a way gives rise to a new movement, in a two-dimensional graph. *Versus* plots an ordered pair of the positions of the two controllers at each time instant t, showing a blue line as a result of this pairing. Therefore, vertical displacement in the graph corresponds to one user's movement, horizontal displacement to the other user's movement. Each user thus contributes to the creation of a *Versus* graph with her own movement, along one of the two dimensions.

This subsection simply wants to leave the reader with a sense of what the software does and the way it does so. Summing up, a main feature is the presence of two (or more) different graphs and users at the same time. In addition, going back to some of the ideas that I have developed in Chapter 1, especially about the way that different instruments engender different ways of encountering (and new meanings of) function, we can also argue that graphing with hand is quite different from graphing with the remote using WiiGraph. At a first glance, this is achieved with a completely different movement, which entail a wider bodily engagement. In light of these main points, we can turn to an overview of the activities with WiiGraph and their design.

4.4.3 Graphing motion(s) with WiiGraph: Moving, comparing, transforming

In this section, we discuss insights about task design from the activities that we carried out with WiiGraph in three teaching experiments in Italian classrooms at grades 4, 7 and 10 (see §4.5). While we recognize that the use of WiiGraph engenders mathematical discourse similar to work with other motion detectors—which have been already investigated in the literature (see again §1.5.2) and which we have also explored (e.g. Ferrara, Ferrari, & Savioli, 2019), we are interested in the ways that we can exploit the potential of WiiGraph through the design of tasks. We believe that this technology fosters novel reasoning about variation and covariation in the context of graphing motion, therefore new ways of exploring mathematical relationships. In fact, the software works with the two remotes, which are to be moved in the same interaction space, in the same time interval. In the meanwhile, there are at least two graphs on the screen, which "move" together while originating in real time on the same Cartesian plane. When two students move with the devices in hand, relationships between movements are at play as well as the relationships between the graphs that are created in real time. Therefore, we can think of the activities as (mainly) unfolding along two dimensions. One dimension is concerned with the (types of) bodily engagement furnished by the technology. The other dimension regards how the concepts of graph and function can be grounded on aspects of covariation, coordination and transformation.

Our main didactical aim was that of creating space in the mathematics classroom for the graphical exploration of functions by means of *Line* graphs, which are produced through bodily movements in space. This was a cross-cutting aim (i.e., common to all grades), and constituted an unconventional (not already known or familiar) way of dealing with graphs for all the classes involved in the longitudinal study, more generally one that does not constitute a common praxis in Italian schools.

In light of the two dimensions above, two main questions were crucial at the design stage. On the one side: What does it matter when I am a body in movement with WiiGraph? On the other: What kinds of graph can I produce (with WiiGraph)? As a way of proposing (one among the possible) articulations of activities, I will guide the reader through a discussion on linear graphs and functions, which brings forth important design choices, as an attempt to illustrate how I envisioned them throughout the design process. First, we already observed that plain *Line* graphs capture the position over time of the two remotes, creating in real time two differently coloured graphs on the screen. Now, imagine two students, who share the same interaction space, while the graphs they produce share the same Cartesian plane. The students who are moving are obviously independent from one another, but we can think of their movements as related to one another. The same can be for their own graphs. For example, the students can stand still for the entire session. Each of them will produce a horizontal, more or less straight, line; the relative distance among the lines will be reflected in the relative positions. Variations in the relationship between the students' positions condition the horizontal straight lines to change their relative positions (one below, above or overlapping the other). In addition, we can think of those graphs as being two elements of a family of straight lines that are all parallel to each other, so we might easily imagine extending the situation to, say, five students or graphs. We can also imagine two different scenarios emerging from the initial configuration of horizontal straight lines: (1) the lines might be stretched in a upward (or downward) direction with the same intensity; (2) the lines might be thought of as collapsing one onto the other and then stretched towards different directions and with different intensities (so that they will have different slope with eventually different sign at the end of the transformation; see Figure 4.5).

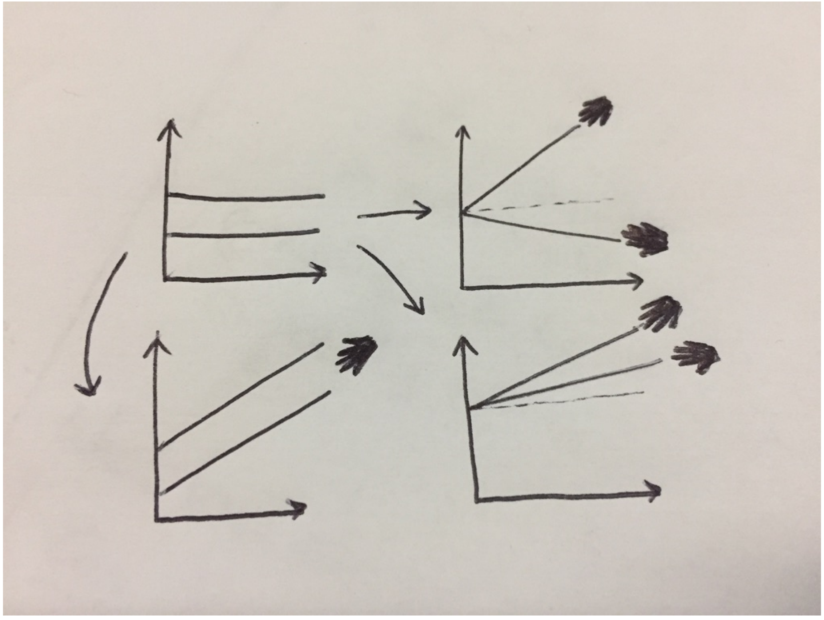


Figure 4.5. My t(h)inkering with transformations applied to linear graphs

Mathematically speaking, both transformations are dilations of the initial lines. However, in case (1) the two lines undergo the same dilation, and being linear functions, they preserve equal distance from each other, while in case (2) they undergo dilations with different parameters and relative distance is not preserved anymore. I can also envision the analogous modifications for achieving the same result with WiiGraph, namely we might ask what it means for the moving students that one of their graphs must result vertically translated with respect to the other when the lines are not horizontal (and even, when they are not straight!). This would mean for the students to modify their previous movements accordingly, preserving distance while being in motion, or escaping towards different directions and with different speeds. In the case of more than two lines, we need to go beyond the use of the software and imagine a new situation, in which students are more than two and, similarly, more differentiated things might occur. We will detail these aspects in §6.5. Here we stress once more a dynamic vision of the graphs and the mutual relationships between graphs' formation/deformation and the corresponding bodily movements' coordination or composition. Thinking of plane transformations that convert the horizontal (parallel) straight lines into slanted parallel or non-parallel straight lines, we proposed that these transformations are to be conceived in terms of the kinds of bodily movements that are necessary to create those graphs. In the current discussion we skipped a crucial point, which directly relates to the fact that using WiiGraph means being a body in movement. The point is that straight lines implicate a constant pace, a rhythmic pattern (always the same for a specific line) to be preserved and perpetuated. These qualitative aspects of graphs creation directly relate to the lived experience of graphing motion, which fundamentally connect to the temporality or duration of the process, since each function in WiiGraph is a function of time, but also because, as bodies, we create space and time in the process of moving (following Sheets-Johnstone, 2011).

Significantly, this example on linear graphs also brings forth the main principles that guided our task design, which I summarise as follows:

(a) a relational vision of graphical representations;
(b) the connection between plane transformations and families of functions;
(c) bodily engagement with a focus on change and relational movements;
(d) imagining beyond the software: being more than two.

These principles will be exploited in the overview of tasks and activities in section §4.5.5, and are entangled with foundational methodological points, which are the core of the next subsection.

4.4.4 Methodology of classroom-based interventions and the teacher-researcher's role

In §4.3 we mentioned the Italian trend in mathematics education research with which our study aligns. In particular, this subsection wants to bring forth the relevance, for our study, of two methodological strongholds of the Italian tradition in mathematics education research, namely the "mathematics laboratory"[3] (Anichini, Arzarello, Ciarrapico, & Robutti, 2004) and the "mathematical discussion" (Bartolini Bussi, 1996).

Italian National Guidelines for the curriculum of both the first and the second cycle of instruction[4] suggest a structure for all the classroom activities that allows for collective discussions and aims to foster the interactions among students, between teacher and students, and with tools, being these technological or not. The National Guidelines also indicate problem solving and group work as important methodological choices in the (mathematics) classroom towards the achievement of abilities in mathematical contexts, reasoning skills and basic rules of social behaviour. All these aspects are beautifully integrated in the notion of *mathematics laboratory*:

> "The mathematics laboratory is not a topic or process area but a series of cross methodological indications, certainly based on the use of tools, technological or not, but mainly aimed at the construction of mathematical meanings."[5] (Anichini et al., 2004, p. 26, my translation)

In this sense, the mathematics laboratory "is not a physical place distinct from the classroom, rather is a structured set of activities"[6] (*ibid*., my translation) which mainly aims at the construction of mathematical meaning with the idea of engaging people (students, teachers), structures (rooms, tools, space and time organisation), ideas (projects, didactic activities' plans, etc.). The idea of the mathematics laboratory encompasses all the situations in which the traditional lesson is modified by the introduction of specific tools and/or modelling activities. Understanding is therefore strictly tied to tool use and to the interactions among learners who work together. This is a specific point that distances this kind of methodology from the traditional frontal lesson. Another quality of the laboratory that has been pointed out, in contrast with the standard lesson, is one that relates to what the word *laboratory* evokes, namely an active and bodily engagement with the matter of study, which is not limited to an intellectual involvement (Paola, 2007). The idea of the mathematics laboratory, which is now shared within the Italian community of mathematics educators as a best practice, has its roots in the work of mathematicians and educators as Perry, Moore, Borel and Vailati's at the beginning of the 20th century.

Another stronghold is the notion of *mathematical discussion*, which was theorized by Bartolini Bussi and colleagues (Bartolini Bussi, Boni, & Ferri, 1995) and is essentially known as one of the methodologies for the semiotic mediation theory (Bartolini Bussi & Mariotti, 2008). Inside that framework,

> "[c]ollective discussions play an essential part in the teaching and learning process where the core of the semiotic process, on which teaching/learning is based, will take place. In a mathematical discussion the whole class is collectively engaged in a mathematical discourse, usually launched by the teacher, explicitly formulating the theme of the discussion. For instance, after problem solving sessions, the various solutions are discussed collectively, but also, it may happen that students' written texts or other texts are collectively analysed, commented, elaborated. Very often, and sometimes explicitly, they are real mathematical discussions, in the sense that their main characteristic is the cognitive dialectics, promoted by the teacher, between different personal meanings and the mathematical meaning related to specific signs (most of the times belonging to mathematics practice)" (p. 17).

Moreover, semiotic mediation theory attributes a crucial role to the teacher (especially concerning the mathematical discussion), who is considered the expert in the classroom environment and guides the students towards the institutionalization of signs, by taking into account the individual interventions of students and by exploiting the semiotic potentialities of the tool in use.

In our teaching experiments, the role of the teacher-researcher is much more that of a questioner than of a guide for the students. She manages the classroom discussions, so she yields to the students one at a time and invites them to speak further whether they are addressing an important issue for the whole class. By priming the activities and prompting the students to engage in meaningful experiments with the software, the researcher values the contributions of the students looking for the main mathematical ideas to be addressed throughout the discussion. Instead, the classroom teacher is an active observer who intervenes in the discussions whenever she feels and helps in classroom management.

We will deepen the role of the researcher in relation to research methods in Chapter 5, and that of the teachers involved in the study as we will introduce them in the next section. An important point for us, concerning the methodologies we used in the classroom, was that the students could freely engage in the discussions and actively partake in the experiments in various ways. This subsection bridges the theoretical aspects of teaching experiments exposed at the beginning of the section with the methodologies implemented for the longitudinal study, which will be discussed in detail in the following section.

4.5 Longitudinal study

The study of my research project involved three different classroom based-interventions carried out in 2017, which were preceded by a pilot study (in 2016). As mentioned in the introductory chapter, the study drew on a previous teaching experiment implemented in a secondary school class of grade 9 students, which was the focus of another research (held in 2014-2015), in which Wii devices were introduced in the mathematics classroom as a way of developing mathematical competencies about graphical representations and the concept of function. We could also benefit of the senior researcher's long experience in designing and implementing teaching experiments with students at different grades, especially with graphing motion technologies in the classroom context (Ferrara, 2006, 2014; Ferrara & Savioli, 2009; Ferrara, Ferrari, & Savioli, 2019). The present project considerably extends the first teaching experiment and constitutes a different set of activities in terms of the purposes and the verticality of the project as a whole, although it relies on the use of the same technology: WiiGraph as a peculiar mathematical instrument (Nemirovsky et al., 2013) for graphing motion, with a slightly different didactical methodology. This section first presents the pilot study, and the participants of the medium-term interventions. The tasks of the study are then discussed and an overview of the days of each intervention is offered.

4.5.1 Pilot experiment

A pilot experiment was held at primary school in a grade 4 class of 21 students (8-9 years old) to test the possibility of using the software with young learners. In the pilot experiment we also explored for the first time some ideas about the development of a longitudinal project focussed on a graphical approach to function with WiiGraph, and specifically on linear functions as a starting point for all the classes involved. The work that was done in occasion of all the teaching experiments also contributed to revise the software with the help of the developers and was useful for the last stages of software's changes and adjustments (see Appendix A for detail).

The pilot experiment consisted of three 2-hour meetings held weekly in the period of May-June 2016 in a School nearby Turin, in Italy. A researcher led the classroom work, which consisted of collective discussions involving the use of the software and group work on written tasks. The students also worked divided in small groups (couples or groups of 4 children) on written tasks prepared by the researchers, during their regular mathematics lessons. The classroom-based intervention constituted the main part of the study. After six months from the end of the experiment, six children who took part in the pilot were also individually interviewed by me. In this occasion, I conducted semi-structured interviews that aimed at understanding what the students remembered from the classroom intervention, with particular focus on their imaginative activity about some specific graphical configurations they had explored through graphing motion with WiiGraph.

The pilot study was significantly relevant to empirically verify whether young students could engage with the software in meaningful ways, without too many technical issues related to the required coordination between the visual outputs produced on the screen and the gestures and actions (with the remotes correctly pointed to the sensor), which are required for creating a smooth graph on the screen. This aspect was remarkable for us because, even if a disconnected graph opens room for a rich discussion on the role of time in modelling motion[7], we are also aware that, especially at early stages, explorations of graphical representation are more easily interpreted whether one can directly relate them to observed movement. In other words, the pilot experiment permitted us to conclude that the kinaesthetic engagement of two children in using the software was not an obstacle to the activity, rather it was possible to successfully and suitably integrate the tool in a teaching experiment with quite young students.

In addition, the pilot served us to highlight some important elements for further designing the tasks of the longitudinal study and grounding the main focus of the research project. We refined our interests in the role of movement by examining the various ways in which a search for coordination in the interaction space was entangled with covariation of functional relationships and graphs' transformations in the Cartesian plane. Most of the issues regarding design have already been exploited theoretically in §4.4.3 and they will be further recovered through the discussion of the tasks given to students.

An emerging aspect, which will be delved in the context of analysis, is the mathematical event of crossing lines. As we will discuss in the following, the crossing of lines captured the attention of students as a peculiar event in the context of graphing with WiiGraph. We will see how its understanding appeared to be pivotal to grasp both the meaning of each single graph and of their relationships (§6.2.2). For these reasons, we exactly took as a fundamental point of the design working in various ways on that particular configuration.

4.5.2 Participants

The study involved three classes, one for each school segment in Italy (primary school, lower secondary school, upper secondary school). A class of grade 4 students, a class of grade 7 students and a class of grade 10 students (of a scientifically-oriented curriculum), all from schools in Turin, were selected. For the recruitments, we contacted the mathematics teachers that are used to collaborate with professor Ferrara in classroom-based interventions and asked them willingness to take part in the study. Among them, we chose to work with two teachers (Tiziana Abbate and Erik Villarboito) from Istituto Internazionale “E. Agnelli” (for the secondary classes) and one (Mariagrazia Trichilo) from Scuola Primaria Carlo Collodi, Plesso “G. Rodari” (for the primary class). The two schools are located in a peripheral area of Turin, where the socio-cultural background of the students is heterogeneous. In their usual classroom practice, the three teachers involve learners in collective discussions and devote moments of their mathematics lesson to group work as it is in the spirit of the mathematics laboratory. Therefore, the students that participated in the study were rather acquainted with working in such ways, even if with substantial differences that I discuss in the following.

The senior researcher and I shared with each teacher the purposes of the teaching experiments and we exposed them our ideas for the interventions as well as the research interests that we pursued with the study. The teachers were informed about the main plan of each day and gave suggestions about the formulation of the written tasks, but they basically made us manage all the phases of the interventions. We asked them to not provide the students with complementary information or activities about graphical representations with respect to those encountered during the experiment, as they had regular classes in-between ours. However, when suitable, they suggested us connections with the topics that they were facing in accordance with the standard mathematics curriculum. The teachers also reported about their class, giving specific information or advice about single students throughout the course of the experiment, whether this was necessary.

The composition of the three classes of the study are briefly summarised in Table 4.1.

Grade 4 **Classe 4F**	26 students (12 females, 14 males)	aged 8-9 years old
Grade 7 **Classe 2E**	27 students (13 females, 14 males)	aged 12-13 years old
Grade 10 **Classe 2A**	15 students (10 females, 5 males)	aged 15-16 years old

Table 4.1. Classes’ composition

When not involved in collective discussions, the students worked in groups, which were formed as detailed in Table 4.2. The groups were heterogeneously arranged according to students' gender and to the teacher's suggestions for each secondary school class, while the children mainly worked in couples, each of them with the habitual classmate.

Grade 4: Classe 4F	13 couples	6 groups of 4 students (2 absentees)
Grade 7: Classe 2E	6 groups of 4 students; 1 group of 3 students	
Grade 10: Classe 2A	5 groups of 3 students	

Table 4.2. Groups' composition

The groups were differently assembled in each class but worked in similar ways (see §4.5.5). The idea that students coming from different school careers and backgrounds, and attending different grades, could engage in comparable "parallel" activities with the same methodology is at the core of this study (as a way to think of a vertical discourse about function in modelling motion).

4.5.3 Verticality of the curriculum

A central issue I want to bring to the fore is the verticality of the curriculum that we designed for the study. In this subsection we will shed light on the significance of this point both concerning our implemented design and with respect to the eventual influence of the study on the curriculum implemented at school.

As a first point, the reader should remember that, while analysing the main ideas about task design in §4.4, we did not mention any pre-requisite that students have to possess in order to tackle the mathematical activities. This is not due to a lack of an *a priori* analysis of the activities. Rather, it was our purpose to leave this partially open for the reasons that we clarify in a moment. It is shared opinion (see e.g., Leinhardt et al., 1990) that foundational ideas rooted in functional thinking somehow traverse the curriculum from primary to secondary school and are crucial for the development of mathematical thinking at all school levels. Therefore, envisioning a spiral didactic – namely one that tends to travel through and encounter the same concepts with increasing depth over the course of the school career – it is highly relevant for educators to think of activities that can be tackled by learners no matter the particular level, and that contribute to the development of functional thinking. Of course, the tasks appear as diverse according to the students to which they are directed. This is especially true for what concerns the language used in the worksheets and the ways in which questions are posed. For example, we expected lower secondary school students to be more familiar with the use of Cartesian coordinates than primary school learners, and upper secondary school students to be more expert on functions and graphs than lower secondary school students. Additionally, we proposed more challenging tasks to older students, also making use of more modalities with them than with primary school children. Apart from this, we addressed very similar tasks regardless of the age of the students, as we will detail in the following (see §4.5.5).

As a last point, in the tasks, great importance was given to the study of movement, that is, to noticing or grasping differences in movement in particular graphical configurations, as an integral and foundational part of the mathematical activity. Movement is not a specific mathematical topic, and indeed might be explored at different times with different degrees of detail, which are not necessarily dependent on a higher competence in (or longer experience with) mathematics as a school topic.

4.5.4 Initial questionnaire

Except for grade 4 students, all the students answered an initial questionnaire, which was not meant to test any pre-requisite, but aimed at collecting information about the students' background on graphs and functions. The opening page of the questionnaire contained the following instructions (for each secondary school class):

> In the next pages we ask you to answer few simple questions. Use whatever you think
> is necessary (words, drawings, …) to be as clear as possible in your answers.
> Remember that there are no right or wrong answers in this case.
> We would like to get to know you with this questionnaire.

The questionnaire was composed of three questions, substantially the same for secondary school students, with just few little differences in the use of words. We present here the upper secondary school version (both versions can be found in Appendix B):

> 1. Have you ever heard about graphs? In which context?
> 2. Explain what a graph is for you.
> 3. Imagine observing two guys who challenge each other in a race: on go, Andrea runs fast, and

Bianca is slower than him. Shortly after, Andrea has to stop to tie his shoelaces. Bianca overtakes him and crosses the finish line first. What would you do for explaining the race of Bianca and Andrea with a diagram?

The third question, in particular, opened room to investigate how the students choose to model a specific situation involving two children, a boy and a girl, who were running in a race. The task was intentionally open to a variety of interpretations, since no reference was made to quantitative aspects of the race, but only to qualitative aspects, and in some sense the information we gave was incomplete for a unique solution. We were in fact interested in seeing how the students approached this situation and the ways that were significant for them to capture it. In addition, in this situation the (piecewise) graphs that model the movements of the two guys (in a space vs time Cartesian plane) would cross each other, and this was a very relevant point to touch on in light of the pilot experiment's insights.

4.5.5 Tasks

The tasks we proposed to the classes were designed and re-touched over the course of the classroom interventions. We distinguish here between different types of activities the students were involved in:

- Experiments (during collective discussions or group work);
- Written worksheets (for individual or group work).

For what concerns the experiments, the researcher asked the students to explore the use of the software through free movements with the remotes, to move for obtaining a specific configuration on the screen, or even to address questions or proposals coming from the class. The experiments were usually performed by couples of students who moved in the interaction space, while the other classmates were watching and sometimes actively partaking (e.g., by giving instructions, standing in space to create reference points for movement, etc.). The researcher led the collective discussion by priming movements in the classroom or addressing ideas that the students brought forth about graphical representations and their interpretation. The different experiments that guided the researcher's work inside the classroom were not entirely predefined in *a priori* design, rather much of the experiments emerged out as a response to specific issues and problems that were addressed in collective discussions. Therefore, we leave this aspect open and we direct the reader to a detailed overview in §4.5.7.

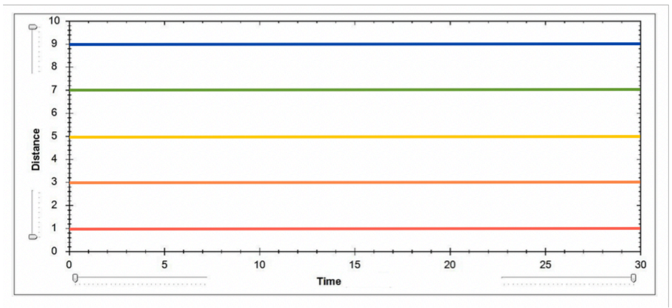
Regarding the worksheets (the entire sheets are in Appendix B), which were proposed to be solved in group or (for secondary school students) individually, Table 4.3 contains the list of the written activities and their main content, also showing the work methodology. The content is exploited right after the table (in the format of the tasks posed in each worksheet, for all grades).

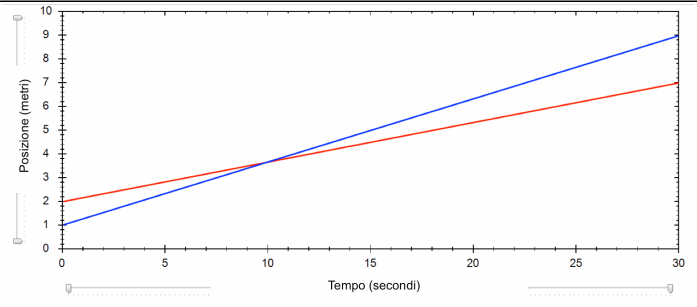
	Worksheet	**Main content**	**Methodology**
Grade 4	Scheda 1	Explain WiiGraph to a friend	Group work (2 children)
	Scheda 2	5 horizontal straight lines*	Group work (4 children)
	Scheda 3	2 parallel slanted straight lines	Group work (2 children)
	Scheda 4	Rob & Bob (v1)	Group work (2 children)
	Prediction	Giulia & Francesca's experiment	Group work (2 children)
Grade 7	Scheda 1	Draw two lines and explain relative movements; try to create the lines with WiiGraph	Group work
	Scheda 2	5 parallel slanted straight lines**	Group work
	Scheda 3	Rob & Bob (1)	Group work
	Prediction	Giulia and Francesca's experiment	Group work (2 students)
	Scheda 4	Rob & Bob (v2)	Individual work
	Scheda 5	Bianca & Andrea's race	Group work
Grade 10	Scheda 1	Draw two lines and explain relative movements; try to create the lines with WiiGraph	Group work
	Scheda 2	5 straight lines***	Group work
	Scheda 3	Rob & Bob (2)	Group work
	Scheda 4	Rob & Bob (v3)	Individual work
	Scheda 5	Sum graph (1st part)	Group work
	Scheda 6	Sum graph (2nd part)	Individual work
	Prediction	Versus (creating a square)	Group work (2 students)
*, **, ***: parallel activities that share the same structure but present a different family of straight lines (v1), (v2), (v3): parallel activities (1), (2): different graphical representations for the same task			

Table 4.3. Worksheets

In the following the written tasks of the worksheets for each intervention are presented. The reader would catch the longitudinal connections between the different grades.

Primary school

Scheda 1 Today we have done some experiments with the controllers. Imagine explaining a friend, who has not participated in this experience, what you understood using WiiGraph, especially about mathematics.
Scheda 2 Imagine that you have 5 controllers at your disposal and that you see these graphs on the IWB:    Explain the way you would create them.
Scheda 3 Last time, we have seen how we can create parallel horizontal lines using the controllers. Now, Bianca and Andrea want to do a new experiment: they want to produce two parallel slanted lines. What do they have to do? For the experiment of Bianca and Andrea, choose two parallel slanted line and draw them. Then, write all the relevant information and advice that you would give to the two children in order to create these lines with WiiGraph.
Scheda 4 Rob and Bob are two little robots that do some experiments with WiiGraph. Imagine entering in the room where an experiment of Rob and Bob just finished. On the IWB you see these two lines:



Tell the way that Rob and Bob moved during their experiment.

Scheda 5

You have observed the way that Giulia and Francesca moved with the controllers.

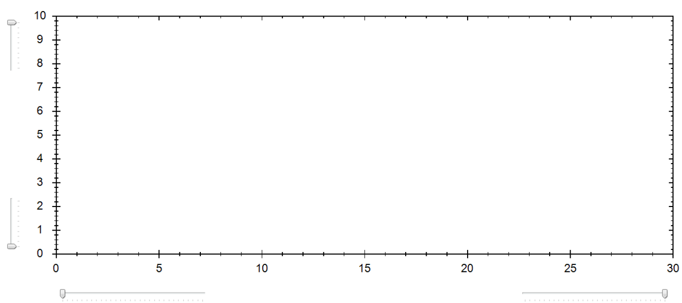
1. In your opinion, which graphs did WiiGraph draw? Imagine these graphs and draw them below: distinguish between Francesca's line and Giulia's line. Use everything you consider relevant.
2. After drawing the graphs, tell why you think that the lines are just those and explain your reasoning.

The tasks of the last worksheet (Scheda 5) were finally given as oral tasks to the students, due to time constraints. They had to predict and draw the graphs associated to the movements they had observed, which were performed by the two researchers (Francesca and Giulia; I put the text of the worksheet for the sake of completeness).

Lower secondary school

Scheda 1

1. Draw two (non-identical) lines:

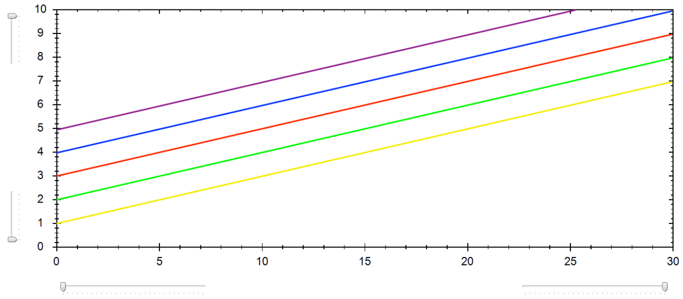


Describe with words the two movements that, in your opinion, provide these lines with WiiGraph.

2. You have no more than two experiments with WiiGraph to try to produce the lines that you drew. Explain how you would modify the movements' description after these experiments.

Scheda 2

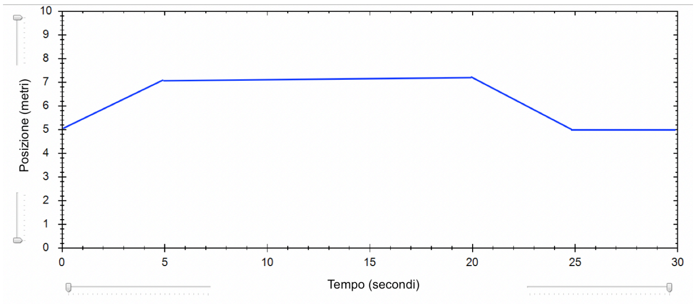
Imagine that you can use more than two controllers and you create five straight lines:



1. Explain which movements can generate these lines, paying attention to specify all the information that is relevant for you.
2. In your opinion, what do the five lines share? Instead, how do they differ? What about the movements? Explain your reasoning.

Scheda 3

1. Rob and Bob are two little robots that have been programmed to move in a very precise way. Imagine that they make an experiment with the controllers together and that WiiGraph produces this line against Rob's movement:



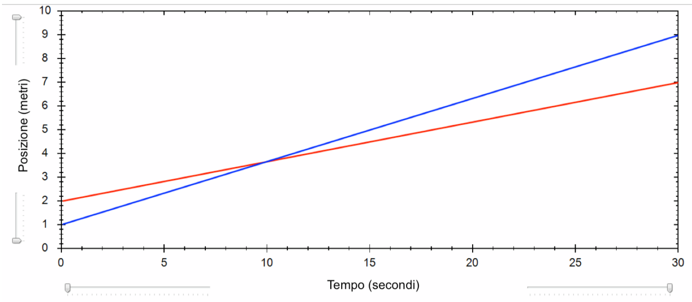
Bob also moved, but its line is hidden!
We only know that Bob started together with Rob, at the same distance from the sensor, but always moved at a double speed and in opposite direction.

- In your opinion, which line would WiiGraph show for Bob's movement?
- Once started, did Rob and Bob meet again?

Explain your reasoning.

Scheda 4

1. Imagine that, with a new experiment, Rob and Bob produced these straight lines:



Describe how Rob and Bob moved, in your opinion.

Did the little robots meet inn this experiment?

Explain your reasoning.

Scheda 5

1. You have already encountered the story of Bianca and Andrea's race:

   On go, Andrea runs fast, and Bianca is slower than him. Shortly after, Andrea has to stop to tie his shoelaces. Bianca overtakes him and crosses the finish line first.

   a. Represent Bianca and Andrea's race using two graphs.

   b. Explain how you reasoned to draw the graphs, adding all the information that you see as fundamental for your explanation.

2. Where do you imagine that the race takes place? Why? Which information about the route do your graphs provide?

3. Draw the route that you have imagined.

Upper secondary school

Scheda 1

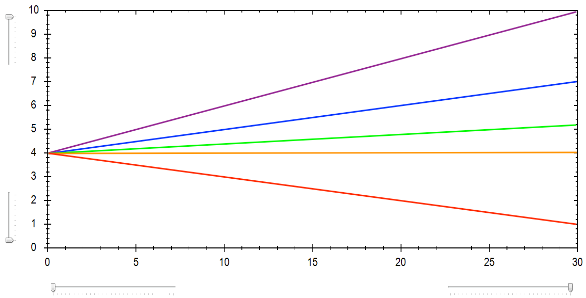
1. Think of two (non-identical) graphs and draw them below.

   Describe with words the two movements that, in your opinion, provide these graphs with WiiGraph.

2. You have no more than two experiments with WiiGraph to try to produce the graphs that you drew. Explain how you would modify the movements' description after these experiments.

Scheda 2

Imagine that you can use more than two controllers and you create the following graphs:

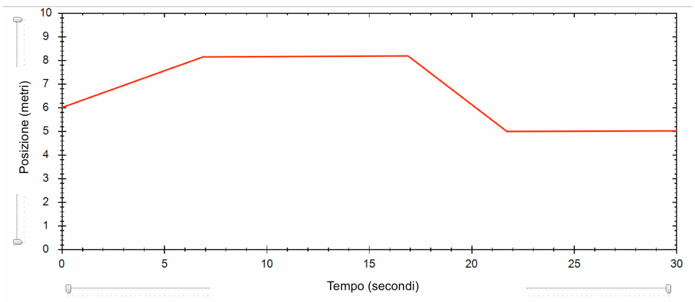


1. Explain which movements can generate these graphs, paying attention to specify all the information that is relevant for you.

2. In your opinion, what do the five straight lines share? Instead, how do they differ? What about the movements? Explain your reasoning.

Scheda 3

1. Rob and Bob are two little robots that have been programmed to move in a very precise way. Together they make an experiment with the controllers. Against Rob's movement, WiiGraph produces this graph:



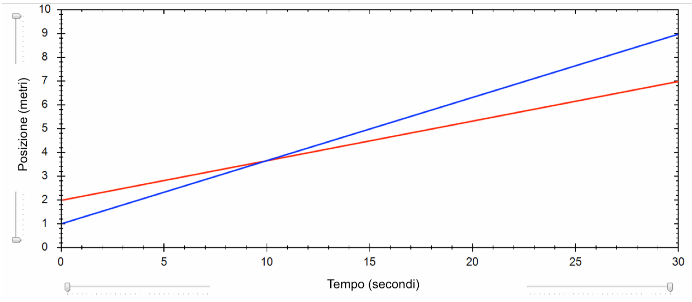
Imagine that Bob moved in this way: Bob started together with Rob, at the same distance from the sensor, but always moved at a double speed and in opposite direction.

- In your opinion, which line would WiiGraph show for Bob's movement?
- Once started, did Rob and Bob meet again?

Explain your answers.

Scheda 4

1. Imagine that, with a new experiment, Rob and Bob produced these graphs:



How would you explain their movement?

Did the two robots meet this time?

Explain your answers.

Scheda 5

1. Imagine telling how the sum functions with WiiGraph to a friend, who does not know the software and never used it. Furnish her:

   a. a suitable explanation

   b. at least one example

   c. at least one suggestion for an experiment

   using everything you see useful and important to be as clear as possible.

2. We have already observed that a horizontal straight line can be created as the sum of two horizontal straight lines.

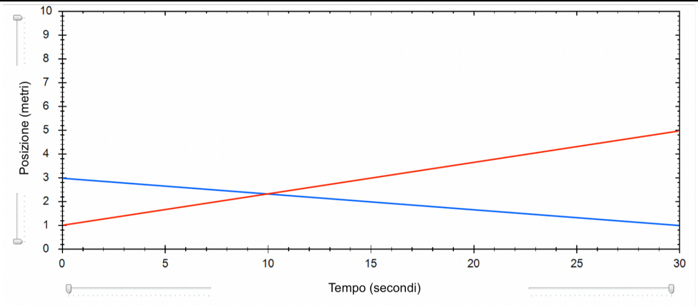
   If we take a particular horizontal straight line, which features do the horizontal straight lines, which allow producing it when summed, have to have?

3. Giulia claims that there exists at least another way, different from the one already seen, to create a horizontal straight line as a sum graph. Is Giulia right? Why?

   Give also some examples to explain your answer.

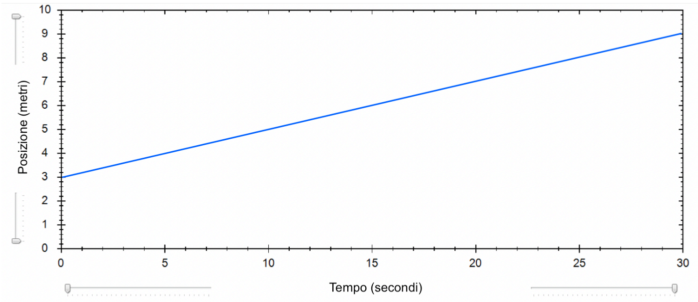
Scheda 6

1. Imagine working with the sum modality and producing these graphs through the movement of the controllers. The sum graph is not visible: How would it be?



Draw the sum graph and explain your answer.

2. Now imagine having this straight line as a goal for the sum:



Suppose you can move the controllers only at a constant speed. With which movements can you create this sum graph?

a. Describe the movements and their analogies and differences.

b. Draw the corresponding graphs.

Explain your answer.

After the end of each teaching experiment, some students were interviewed as already mentioned. Structure and aims of the interviews are detailed in §5.1.6, as I do not consider them as integral part of the classroom-based interventions, rather I intend them fundamentally as methodological tools for the research.

4.5.6 Final examination

The lower and upper secondary school teachers wanted us to prepare a final examination to be proposed to the students at the end of the intervention. The final test was evaluated by the teacher himself, in the case of lower secondary learners, while the grade 10 teacher asked us to assign marks (A, B, C) to her students. The questions of these examinations are entirely shown in Appendix B, and will be translated in the text, where necessary.

4.5.7 Overview

In this subsection, I offer diagrams for the (*a posteriori*) emerging structure of the three classroom-based interventions (Figures 4.7, 4.8 and 4.9). The legend of the structure is to be read in Figure 4.6. From this overview it clearly appears how the various work methods and moments (discussions, experiments and individual/group work with the tasks) were amalgamated and balanced for each intervention.

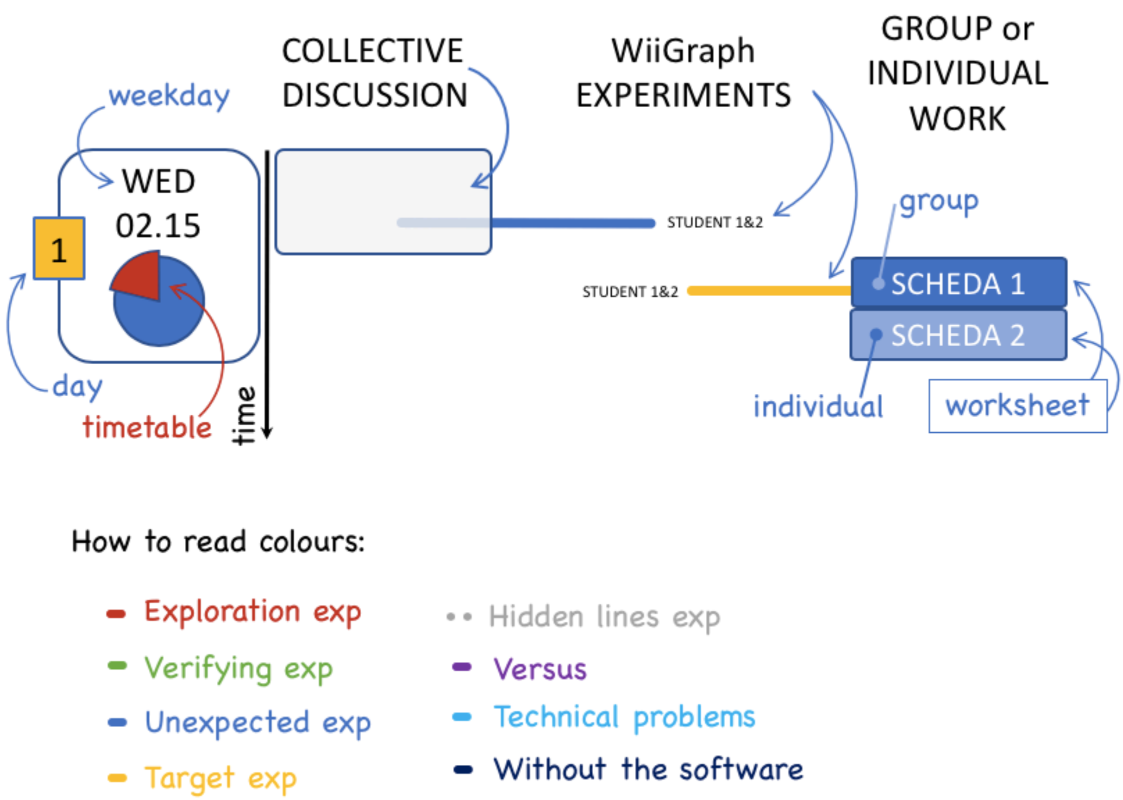


Figure 4.6. Legend for the diagrams of the teaching experiments

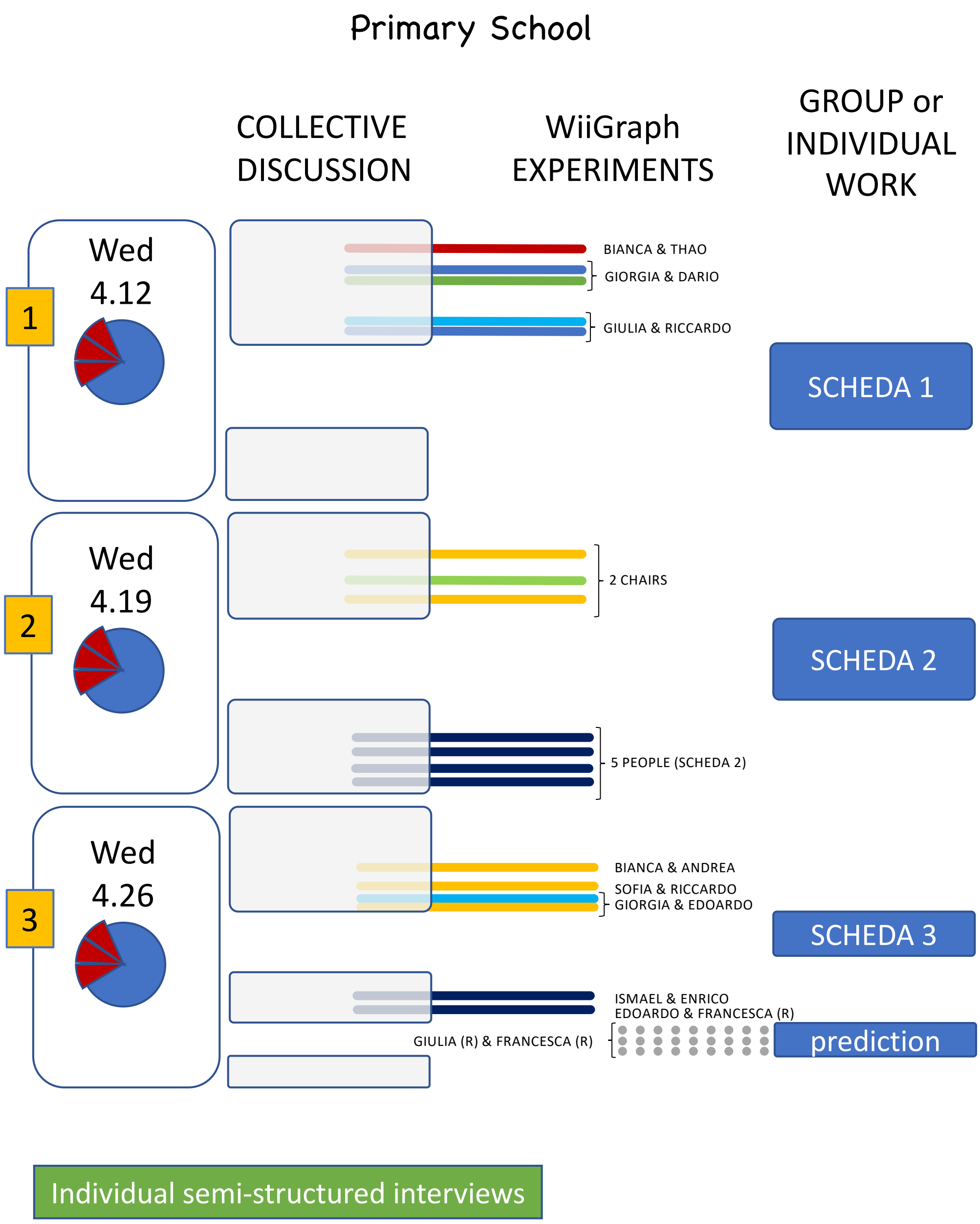


Figure 4.7. Diagram for the grade 4 teaching experiment

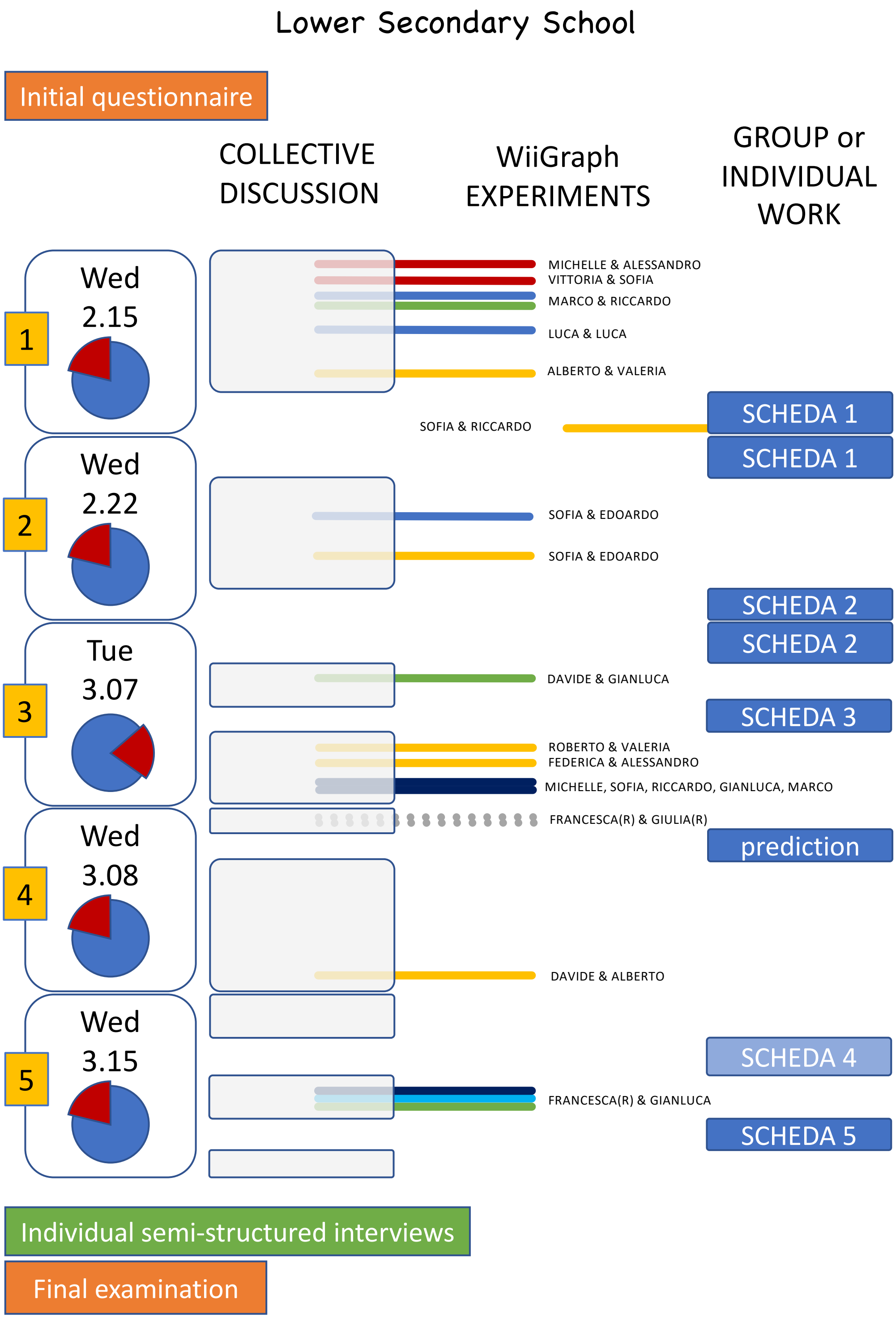


Figure 4.8. Diagram for the grade 7 teaching experiment

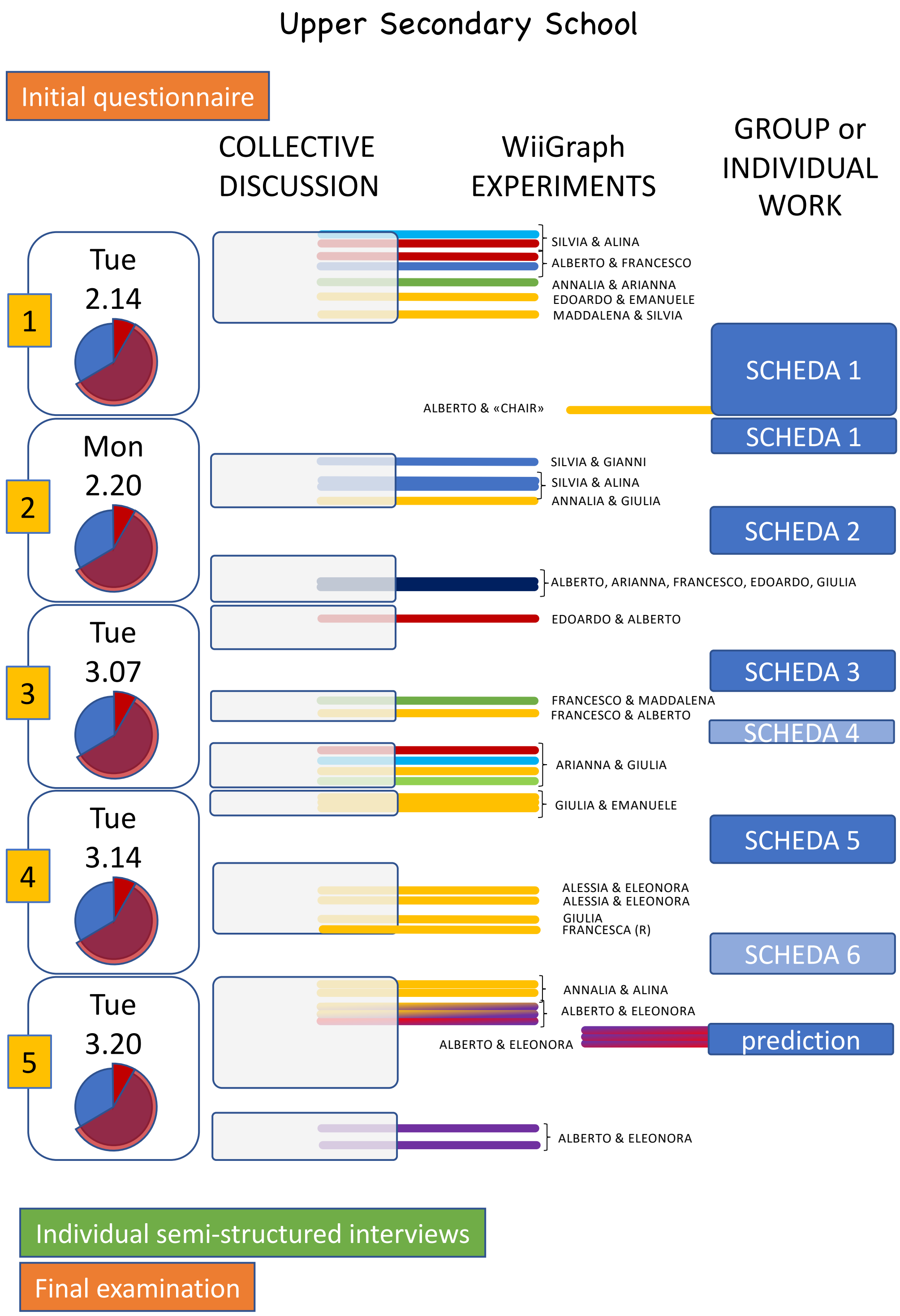


Figure 4.9. Diagram for the grade 10 teaching experiment

[1] As already anticipated in the Introduction, WiiGraph was developed by R. Nemirovsky (Manchester Metropolitan University) and some colleagues (C. Bryant, M. Meloney and B. Rhodehamel) from the Center for Research in Mathematics and Science Education of San Diego State University (see also Appendix A).

[2] A first discussion on our methodological approach in the mathematics classroom is to be found in Ferrara and Ferrari (2017c). Specific attention is drawn to collaborative mathematical tasks with WiiGraph in de Freitas, Ferrara and Ferrari (2018).

[3] *Laboratorio di matematica*, in Italian.

[4] First cycle: "Indicazioni Nazionali per il curricolo della scuola dell'infanzia e del primo ciclo d'istruzione" (MIUR, 2012). Second cycle: "Indicazioni Nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali" (MIUR, 2010).

[5] "Il laboratorio di matematica non costituisce un nucleo di contenuto né uno di processo, ma si presenta come una serie di indicazioni metodologiche trasversali, basate certamente sull'uso di strumenti, tecnologici e non, ma principalmente finalizzate alla costruzione di significati matematici." (Anichini et al., 2004).

[6] "non è un luogo fisico diverso dalla classe, è piuttosto un insieme strutturato di attività" (Anichini et al., 2004).

[7] Using WiiGraph entails the recognition that a "hole" in a graph means that the remote has not been pointed to the sensor during the graphing session. This does not usually obstacle holistic interpretations of the lines, rather often opens discussions on the fact that holes are formed since the modelling time runs no matter what the user does. In case the software cannot produce the graph(s) apart from small traits of disconnected lines (e.g. because of bad light conditions, which obstacle the infrared technology, or because the remotes are not correctly pointed to the sensor bar), to interpret the produced representations is very difficult or partially misleading.

5

Research methods

For this project we conducted a video-based study of the three teaching experiments (detailed in §4.5), which comprise classroom discussions led by the researcher, experiments with the software and students' group work. At the end of each teaching experiment, I interviewed some of the students that participated in the study. The collected data include the video recordings of classroom activities, informal semi-structured interviews with some students and written protocols coming from the individual and group work of the students. A master student also took part in the teaching experiments held at primary and lower secondary school. This chapter will detail the methodology of the research project concerning data collection and analysis. A brief discussion of the quality of the results will highlight the ethical concerns of the qualitative inquiry I pursued for the present study.

5.1 Data collection

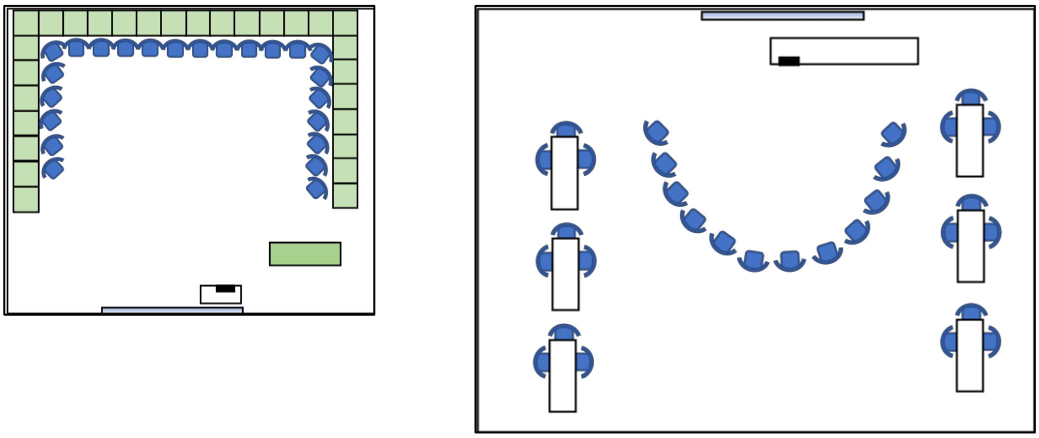
In Chapter 4, I detailed the structure of teaching experiments and discussed the activities as well as the emergent structure that characterised the classroom interventions. The complexity of classroom contexts embraces a variety of situations and interactions, which is materially impossible to capture in its entirety. In this study, I employed the main methods of data collection that are commonly used in the field to capture embodied interactions and mathematical activity of students, that is, video recordings of classroom activities and collection of students' written protocols (answers to assigned worksheets as well as their diagrams, especially those that emerged from tasks of graph prediction). When possible, I also took screen captions of the software's sessions. The data collected additionally consists of video recordings of the interviews with some of the students involved in the project, and the students' diagrams that emerged out of discussions with the researcher. Video data is in fact used throughout the learning sciences as a common form of documenting learning events (Nemirovsky et al., 2018). The mass dissemination of recording technologies has been pivotal for the emergence of innovative research methodologies in the social sciences (de Freitas, 2016b; Pasqualino & Schneider, 2014). In my study, video data is particularly relevant in relation to the interest in studying the human body in the mathematics classroom, where a multiplicity of bodies interact and are in movement. In this section I develop methodological issues concerning data collection for the present study. As a starting point, the verticality of the study design (see §4.5.3) is also mirrored in the methods of data collection. The collection was basically the same in every intervention. Small differences were due to material constraints (like the setting or environment) or institutional policies. We will discuss these aspects for each teaching experiment. When not further specified, what is said directly refers to the study as a whole and is therefore applicable to each teaching experiment. The following subsections describe the strategies used in order to capture students' activity inside the mathematics classroom.

5.1.1 Classroom setting

The school that hosted the activities for the grade 7 and grade 10 interventions made available a big laboratory room, usually used for mathematical activities, in which we could work for the whole duration of the experiments. In such a room, the students were sitting around a wide space, facing the Interactive White Board (IWB) that projected the computer screen to the entire class. For group work, the students moved and sat around big tables joining their group mates.

The classroom preparation involved the positioning of tables and chairs as shown in Figure 5.1b, which allowed for creating, in front of the IWB, the interaction space devoted to experiments with the software. Since the room was not the usual classroom in which the students carried their activities, we prepared it before the students arrived for the intervention. This gave us the possibility of managing collective and group work within a relatively ample space on the side, so that the different groups were not disturbing each other. It also allowed creating enough space in the middle of the room, so that all the students could look at the IWB and, at the same time, couples of students could freely move inside the room. This aspect might sound not so relevant, but in fact is highly significant since the experiments with WiiGraph and the embodied interaction in general were a crucial part of the study and its design.

At primary school we instead worked within the regular classroom, even if we changed the usual disposition of desks and chairs, creating a similar working space, in which children could move and collective discussions could occur in a way that the students could face each other, as well as the IWB and the researcher at the same time, differently from the standard classroom setting; see Figure 5.1a).



(a) (b)

Figure 5.1. (a) Classroom setting at primary school; (b) classroom setting at secondary school

Two cameras were positioned in the classroom during collective discussion: one usually directed towards the IWB screen in order to capture the real time graphs' production and, at times, the students who came to the IWB for explaining their reasoning (e.g. pointing to, or moving around, the surface); the other camera was used as mobile and recorded the students while moving in the interaction space or the students-researcher interactions.

During group work, one camera was fixed on a focal group while all the members were dealing with the written tasks or two were interacting with the software, while the other camera followed – time to time – the work of other different groups (and, sometimes, moments in which the students were informally interviewed regarding their group activity).

5.1.2 Video recordings of classroom activities

The main data for the research study is exactly given by the video recordings of classroom activities throughout the interventions, based on the use of the two cameras.

In the very first teaching experiment with WiiGraph (see the introductory chapter), we could only rely on a single video camera to capture classroom interactions, and we experienced considerable difficulties in recording embodied interactions with the software (students' bodies in movement), the software window and the engagement of the rest of the class (students and researcher) at the same time. Due to this previous experience, and since we could use a new version of the software that allowed built-in capture of graphs and replay and recording of graphing sessions, we tried to improve our methods of data collection, especially concerning video sources and graphical outputs created with WiiGraph.

Nevertheless, there were also constraints that limited our freedom of action. We used the software, which still consisted of a beta version, with a relatively old personal computer that, for example, did not allowed automatically recording the software's window. Sometimes, the quality of the video recordings was influenced by poor light conditions, after that shading the classroom was necessary to avoid light interferences with the infrared technology. Additionally, in the analysis phase I realised that other viewpoints could have been useful to map the embodied interactions with the software, for example a top view of the classroom floor. For practical limitations, this was neither possible nor designed at the beginning of the study, so we might discuss these aspects only in relation to possible improvements for future research.

Additional data collected in the interventions are detailed in the next subsections.

5.1.3 Initial questionnaires

Except for grade 4 children, before the beginning of the intervention, the students answered an individual questionnaire, which was delivered by the classroom teacher, without any specific instruction (see also §4.5.4). They could use words and drawings, as they felt comfortable to.

The questionnaires were collected and scanned before the experiment and gave us insights about students' previous knowledge of graphs and functions.

5.1.4 Students' written protocols

We collected the written productions of the students, from the individual and group work on written tasks, as well as the diagrams produced during collective discussions (for example at the whiteboard). The worksheets were picked up at the end of each day and then scanned, no matter whether they were completed or not.

During group work, the groups filmed with the cameras provided additional information about the students' interactions that led to the written protocols and particularly to diagrams. The final examinations also gave materials to be analysed in terms of both thinking processes and achievements, in particular in relation to the individual learners. Indeed, this was a real source of information about students' change across the intervention.

5.1.5 Digital screen captures and WG files

When possible, we saved digital screen captures of the experiments with WiiGraph, which consist of file images of the final graphs created by the students using the remotes. Usually, the projected computer screen was recorded by one of the available cameras, from a side view. In some occasion, we could save the experiments in a special format, which allows for the replay of the experiment. The replay was especially used during collective discussions to verify conjectures that the students brought forth, or, during the analysis phase, by the researcher, in order to compose the movements and the dynamic graphs (see § 5.2.2).

5.1.6 Interviews data

I conducted individual semi-structured interviews with some of the students that participated in the teaching experiments, in order to widen our understanding of their thinking processes, as well as to shed light on the interests fuelled by the intervention.

Semi-structured interviews incorporate both open-ended and more theoretically driven questions (Galletta, 2013). The questions aimed at eliciting data grounded in the experience of the participant as well as issues more entangled with our research interest. The senior researcher filmed the interviews with a mobile camera. In this occasion, the interviewer and the interviewee sat around a table, and each student had at her disposal some blank sheets of paper, pens, the two remotes and the computer running WiiGraph.

5.2 Data analysis

As already mentioned, the collected data were used for a qualitative analysis. De Freitas, Lerman and Parks (2017) analyse qualitative methods in mathematics education and observe that in some sense, in education, all data is qualitative "in that it pertains to the lived experience of humans as they participate in various educational processes" (p. 160). The authors push this discussion further by noticing that qualitative methods are attentive to capture aspects of data concerning educational context (no matter what kind of data), which might end with being overlooked when the research practice is too narrowly oriented towards coding and quantifying the processes or behaviours that are under study. Stinson and Bullock (2015) discuss change in paradigms of inquiry in mathematics education history and how this is reflected in the relatedness of methodology, theory and analysis. They observe that "the most significant consideration for mathematics education data collection within a critical postmodern paradigm is destabilizing the researcher-participant dyad" (p. 12), that is, making the object of inquiry the focus of data collection, without imposing the point of view of the researcher. This implies to resist claims of authority and to understand that the particular researcher's account is only one among many. Moreover, according to the commonly shared vision of mathematics education experience as dynamic and complex whichever the settings or the conditions (Skovsmose & Borba, 2004), researchers in the critical postmodern phase use many media and various form of data representation to communicate their results and approaches in more accessible and mobile ways.

In this study, I pursue a qualitative analysis of data, relying on the ethnographic tradition (Eisenhart, 1988), attending to bodily interaction not only as multimodal but also multisensory (Streeck, 2013) and using micro-analytical tool to capture the nuances of the perceptuo-motor-imaginary activity of the students (Nemirovsky & Ferrara, 2009).

5.2.1 Data preparation

The collected written worksheets were examined after the intervention days. At the end of each intervention, I then organised the video data in tables, which summarised the main content of each video. Watching video material was the first step to highlight emerging issues, which then guided the analysis. Throughout, I have also designed and produced the diagrams proposed in Chapter 4 (see §4.5.7) in order to highlight a structure and the particular way of working inside the mathematics classroom, which characterised the teaching experiments.

5.2.2 Video analysis

Most of the analysis I carried out is devoted to capturing and tracing the moving bodies and choreographies of collective movement in order to better grasp the potentiality of the individual and collective body as a center of indeterminacy and understand dynamic aspects of temporality as duration. Following de Freitas (2016), in the style of early scientific cinema, the assemblage of the data helps us examine the mathematical event and the entanglement of mathematics and the learning bodies, as numerous unanticipated contingencies get incorporated. In so doing, we hope to offer a vision of the body primarily as an *expressive* body. The actions of such a body are not mere communicational and cognitive representations of rational thinking but are an actualisation of the qualitative kinaesthetic dynamics and "gradient information" (Sheets-Johnstone, 2011) experienced by students through change. This perspective calls for the development of experimental methodologies to enrich research practices based on video recording and the subsequent use of professional video editing software (Derry et al., 2010). As an example, during video analysis I used a Multicam Editing Software (Final Cut Pro), which allows for automatic pairing of video sources that have been recorded simultaneously from different angles. The software works through audio synchronisation, which uses audio waveforms to compare and match different sources over time. Therefore, it creates video displays that (through the audible) embrace multiplicity of points of view around a learning event, assembling the researcher and the digital in new ways. Here I want to put the emphasis on the process of assembling of the researcher with the data through a discussion of how synchronised multiple video streams help us: (1) make apparent distributed and unexpected dimensions of the classroom event; and (2) re-assemble complex learning events which involve a multiplicity of bodies simultaneously active in the classroom.

The integration of videos from multiple sources may question the very act of seeing, interpreting, and learning of students, educators, and researchers. In addition, it addresses current issues emerging from theories that portray bodies as dispersed across auditory, visual, digital, kinaesthetic, and material dimensions (e.g. de Freitas and Sinclair, 2014;

see also Intermezzo: Inclusive materialism). In “Matter and Memory”, Bergson’s (1896/1988) argument begins by first carefully examining the distinctive nature of perception and recollection. Perception, he argues, is first of all impersonal and diffused. If we bracket for a moment the way that memory infuses perception with personal meaning, we will begin to grasp how perception is a material encounter between bodies rather than an act of coming to know (de Freitas & Ferrara, 2015). Rather than reduce perception to the hand-servant of knowledge, Bergson asks that we consider perception as a material encountering rather than only a source of knowing. Perceptions are not pictures or photographs taken of the world and stored in the brain, but are rather constituted in the material relations that sustain encounters between bodies. Perception is therefore less about the creation of representations in the mind, and more about this material tension that couples or assembles bodies with other bodies.

In particular, when I take two points of view and I join them through the auditory dimension (via editing software), assembling the digital is more about working with the composition of the two videos instead of their simple coordination.

From a technical point of view, the shapes of sound-waves are matched by the software to synchronise the two video-clips obtained with each camera. In Figure 5.2, we can see the superposition on each line where sound-waves are depicted.

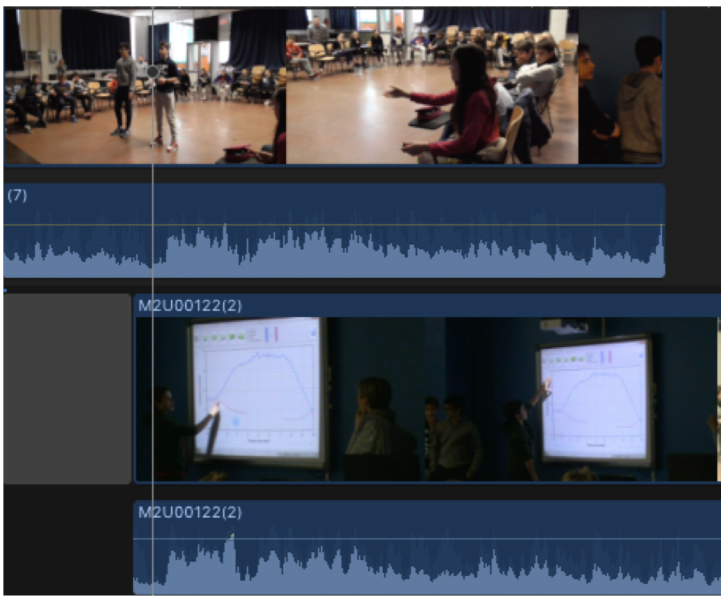


Figure 5.2. Two video sources matched inside the editing software (Final Cut Pro)

In some sense, it is of interest here that matching the perspectives comes from matching symbolic representations that speaks of the subterraneous dimension of digital images, that is, audio. Thinking about how the digital interlaces with the work of the researcher also opens up questions on how this permeating substance, sound, is actually the glue that sustains the complexity of the event. In line with de Freitas (2016), the engagement of the researcher with the composed data is not a question of seeing more, or better, but to *look at* the event differently.

In our particular setting of technological tools and moving students, I argue that three main questions might be significant with respect to this line of thought: (1) questioning who is the principal agent (Is it the human body? Is the technology just giving feedback?); (2) re-building the point of view of observing students, who look simultaneously at the two moving images; and (3) re-thinking the learning event from the point of view of students who are moving and that of those who are not moving.

It is the auditory dimension that grants us the possibility to join the different points of view (namely the dimension that is not prominent in the two moving images, even though it might become prominent sometimes, e.g. when the teacher asks for silence, or when, as the experiment stops, the students all laugh). Assembling with the data throughout these questions is in line with the theoretical commitments and bring forth discourses about perception as diffused and impersonal by looking at how the event is distributed (throughout spaces, relationships, affects, etc.), and at the unexpected and new (regarding lines, speeds, etc.). In so doing, I also aim at deepening how bodies are dispersed in-between spaces and points of views, and get together creating a collective memory of the classroom (de Freitas & Ferrara, 2015), while showing how movement is not only a way of verifying conjectures but implicates new possibilities and explorations. Furthermore, rethinking graphs and movement in such a way opens up the possibility of thinking of the way that the body can be seen as a notation for the graph. We delve into this issue in the next subsection.

5.2.3 Looking for a movement notation

Arzarello (2006) proposed the methodology known as “semiotic bundle” to enlarge the semiotic system considering also gestures, glances and extra-linguistic modes of expression as relevant to the understanding of learning processes. In fact, the semiotic bundle takes into consideration the signs supposedly relevant if we consider mathematics (teaching and) learning as a multimodal event. The paradigm of multimodality mostly developed around the 2000s, fostered by theories of embodied cognition and the studies on mirror neurons. The great connection that the semiotic bundle approach played in our field was that of showing the importance of the multimodal nature of mathematical cognition, beyond the focus on linguistic elements. While, on the one side, this accentuated the role of bodily activity in mathematics, on the other, it stressed the relevance of discourse on ground-breaking conceptual metaphors that are part of mathematical conceptualisations, according to the theorizing of embodiment (Lakoff & Núñez, 2000).

In my work, I share with the researchers who followed this line of research, the interest in, attention to and recognized relevance of bodily movements and actions with the mathematical, with some theoretical difference that also imply difference regarding the chosen methodology. First, my account for bodily movement does not only reside in the cognitive studies that try to highlight the fundamental role of bodily experience in mathematics. Indeed, this is crucial in my study but is not the only point that matters (see also Intermezzo: Movement), what brings forth a methodological question: How to study body motion as, in fact, more than just motion, as movement? More precisely: How to speak about the experience of moving with WiiGraph in a way that it does not (in the first place) deny the very complex nature of movement?

Our main aim both from the point of view of the didactical interventions and of the methodological standpoint exactly was *to put the body at the centre, without making it the centre*.

In other words, we want and attempt to give the body its relevance concerning what the body is capable of, its potentiality, the nature and indeterminacy of the body, without falling into considerations that all the activity springs from the body, its will of acting or its own agency. This resounds with the issues raised by our theoretical commitment to movement (see Chapter 3).

Since the very first analyses it was apparent that I could not simply describe the students' movements in terms of changes in position. The whole experience of movement also included little events such as laughter, turning heads, bending knees, resolute steps and intense stillness. I could not ignore all of these micro-aspects and I felt the need for looking for ways of discussing the qualitative dimension of movement and integrating them into the methodological tools for the analysis. I noticed that there were analogies between movements and (dance) choreographies. If we think of a dance as bodily movement according to a rhythmic pattern or music, within a given space, we can grasp some relationships with our motion experiments. In fact, the experiments with WiiGraph always required more than one person moving in the interaction space. Moreover, talking of an experiment requires coordinating two people's movements in a shared space-time. One-dimensional motion is expected to be performed but there is much more freedom in movement, especially when the students explore the use of the software for the first time. The body positions and movement are captured through the software by a remote that *enlarges* the body and *expands* its potentialities to mathematical representations and meaning. All these aspects resonate with a vision, like that of Manning (2012), which claims that that which is crafted choreographically are not bodies but relations. Certainly, we cannot consider a motion experiment fully as a dance if we think of the purposes for which dance and experiment might take place, but there still is an essential, common point: talking of an experiment implicates the coordinated movement of two people (a coordination both in space and time).

Particularly helpful was in this sense the discovery and study of dance notations, an element that seemed to give the conjunction point between movement and the production of diagrams for speaking about it. According to Youngerman (1984), "notational systems are more than tools for documentations; they are systems of analysis that can be used to illuminate many aspects of the phenomenon of movement. Notation scores embody perception of movement" (p. 101). There exist several types of notation or movement notation, which may be not directly related to dance in particular, but to the recording of human movement more generally. In dance, a notation is the translation of four-dimensional movement (time being the fourth dimension) into signs written on two-dimensional paper. Dynamics, or the quality, texture and phrasing of movement might be seen as a fifth dimension, but this is not an integral part of every system. Youngermann (1984) reports on three major systems of movement notation: Labanalysis, Benesh Movement Notation, Eshkol-Wachmann Notation. Labanalysis is the most widespread and is composed by: (a) Labanotation and (b) Effort/shape. Labanotation is a notation system for the description of the structure of movement. It deals with the "what", "where" and "when" of movement. Briefly, in Labanotation a sequence of signs describes the choreography: the shape of the basic sign indicates the direction of movement, shading shows the level of movement (high, middle, low), relative length expresses time duration of movement. Placement of the sign on a three-line vertical staff indicates which is the body part moving (see Figure 5.3a). The central columns contain symbols that record the transference of weight, therefore, by default, they refer to the movement of left and right leg (foot). The other symbols placed on the right and left of the staff are used to capture gestures in a movement (arms' and head's movement). At the bottom of the staff the initial position might be depicted, as we will explain in the following.

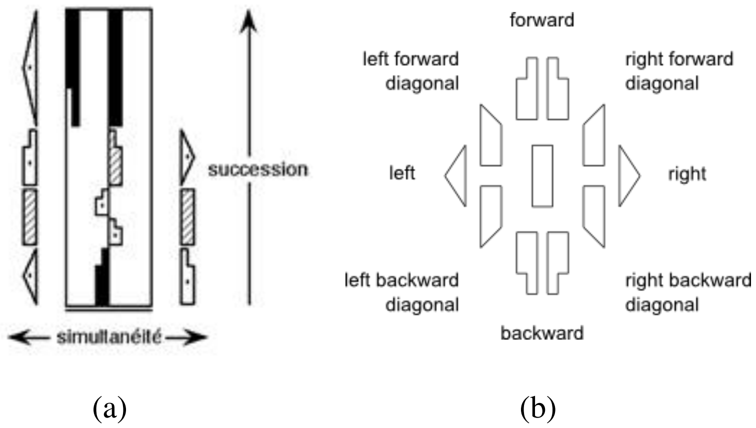


Figure 5.3. (a) Example of Labanotation staff; (b) direction of signs in Labanotation

Reading the staff vertically reveals succession in time; reading horizontally reveals inter-relationships of body parts at one point in time. Thus, direction, level and timing of an action are captured in one symbol (Figure 5.3b). Effort/shape system instead describes the quality of movement, the "how" or adverbial dimension of moving.

5.2.4 A tailor-made movement notation

To take into account the complexity in/of the moving body means to highlight micro-movements and micro-perceptions that constitute the wholeness of a movement, to open discussions around that which is intentional, that which is not, that which is worth of analysis, that which could not be stressed or highlighted without reducing movement to motion. Positioning with respect to such an understanding of movement assumes the nuances of both an act of responsibility with respect to data and a methodological stance concerning data and the people who partake in movement. One main issue is to consider movement not just as an action or a sequence of actions, and instead turning to a conception for which there is only movement as such (becoming) without any classificatory aim. In this research, an understanding of movement is also crucial to highlight *what the movement experience is like for the students* in the context of motion experiments. Differently from the majority of research in mathematics education, where hand gestures, facial expressions or gazes are the main "type" of bodily movement taken into account, in this study I felt the necessity to take into account the whole-body movement happening at the core of the motion experiment. For the analysis of movements and experiments from the mathematics classroom, I have therefore created a modified version of Labanotation for the purposes just described. Using the movement notation, my purpose is to keep trace of, and touch on, the petite happenings that move the body and that helps qualitatively characterise the experience of the experiment.

In particular, the movement notation that I refined for the sake of analysis mostly draws on main ideas and structure of Labanotation. It aims at capturing the movements of the students in the interaction space while they use the software or, sometimes, pretend to use it. The basic symbols that I use are shown in Figure 5.4: they capture respectively a step forward and a step backwards in the interaction space, meaning a step forward as directed towards the sensor bar, and a step backwards as performed while facing the sensor and the screen but moving in the opposite direction. The symbols appear with a middle level shading of Labanotation (a point in the middle of the sign), since the students normally step back and forth without, say, jumping.

The symbols in Figure 5.4 are extracted from a particular staff and only refer to the right side of the body, and to a linear direction of movement as the reader can spot from comparing them with the signs presented in Figure 5.3b. Instead, stepping forward usually means alternating right and left steps.

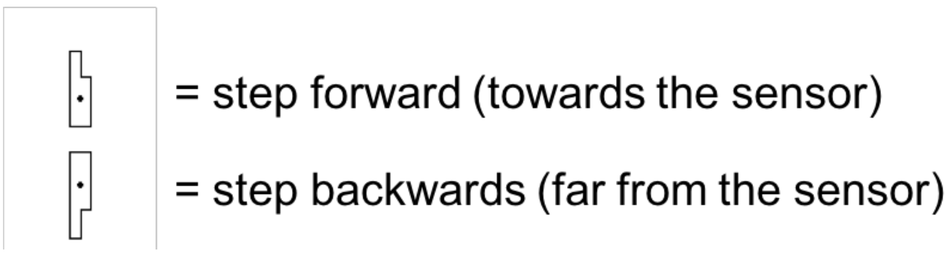


Figure 5.4. Basic signs for the movement notation

A notation like the one offered in Figure 5.5 therefore captures an experiment in which two people are respectively stepping forward (A) and backwards (B) with respect to the sensor. All the symbols have the same length meaning that all of them take the same time to be performed, implying that A and B move more or less at the same speed.

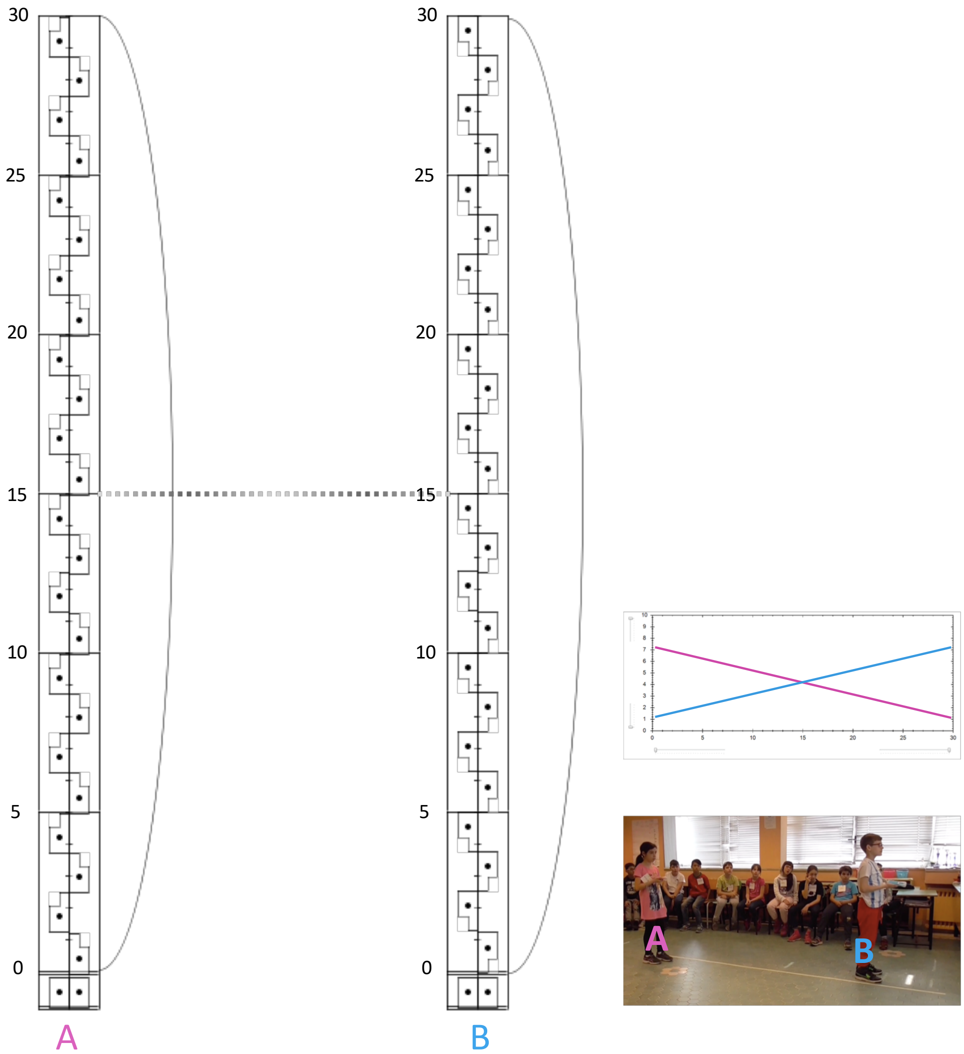


Figure 5.5. Hypothetical choreography involving a couple of students in the creation of two straight lines with opposite slope in WiiGraph, with corresponding graphs and initial positions

The line on the right side of each staff wants to stress continuity of movement, which might characterise movement performed without gaps or discontinuity (mostly uniform). The dotted line that connects the two staffs indicates that at that time (15 seconds) A e B are at the same distance from the sensor. Other symbols might be added when necessary to highlight gazes or peculiar aspects of movement, and they will be introduced when necessary in this dissertation

Next to each staff, the numbers correspond to the time interval visible in WiiGraph, further divided into short intervals 5 seconds long. The strokes inside every 5-second interval indicate each 1 second.

In addition, instead of using the floor plan that shows the initial relative positions of the two users, which are crucial to understand the movement with respect to the surrounding space, I have chosen to present the notation together with frames of the original video from which it is extracted. Other frames also complement crucial moments during the experiment, which should help the reader grasp the relationship between notation and the movements.

All the notations that are offered in this work have been realised using a free application named LabanWriter[1], which allows for creating movement notations utilising the symbols of Labanotation.

5.2.5 Discussion of a hypothetical movement notation

A point which was raised by the examination of a hypothetical movement notation for an experiment like the one shown in Figure 5.5 is how the notation *might engender a topological understanding of movement* within an experiment. Creating lines that cross each other in a symmetric form, or that are parallel to each other, are events in that they are specific instances in an entire bundle of possibilities that collects every couple of lines that somehow comply to that visual arrangement. A configuration of this or that type can be expressed through a notation that would recreate two graphical representations subordinated to that particular request or constraint. We can see that in each bundle, those couples might be conceived as obtained one out of the other from particular transformations applied to all the symbols in the notation, e.g. the shrinking of the symbols in each staff. Moreover, relationality of two movements is preserved whether each staff is subjected to the same transformation, engendering a sense for which the deformation of the symbols via a transformation changes the quality (e.g. the speed) of the movements but not the relationships between the two corresponding graphs.

This is only true from a theoretical point of view (for example, in the case of the hypothetical thought experiment above), and of course speaks directly to a vision of experiments which is not able to capture their indeterminacy and instability.

At the same time, this is not the only aspect which can be further discussed theoretically. I offer two other points that are worth mentioning here: (1) the interplay between discrete and continuum and (2) the illusory removal of the human body from the picture.

Regarding the first point, the notation aims to capture the movement as a continuum while crystallising it into a sequence of standstills. The interplay between discrete and continuum lives in the notation and pushes forward in this methodological approach discourses about the continuity or flow of the experience versus the gaps of perception as problematic in a deep understanding of movement.

On the other hand, we may ask ourselves: Is this method erasing the human body from the picture? My tentative answer is *no*: this is rather a way of capturing certain qualities for the human body, which is also one of the goals for which the choice of a notation came up. Indeed, the notation that I am presenting should constitute an enlargement of our understanding of what the graphical notation says about the flow of the students' movements in space (in an experiment).

Each element in the staff does not really captures speed but the whole progression give a sense of variations and changes in movement, like the rapidity at which the step occurs in relation to other steps close (in time) to it. We cannot have a sense of the exact speed by looking at the symbols since they do not take into consideration the length of the steps, but only the time that it took to perform that step by means of their own length. What we can instead get from looking at contiguous symbols is the possibility of changes in speed within the whole movement or sudden steps, which occur (big or small) in a brief interval of time. This way of looking at the notation is pointing more to movement as change than movement as a sequence of actions, even though one might argue that the notation is capturing in the first place a sequence of discrete moments within the movement.

More importantly, in the movement notation, reference to the distance from the sensor bar is lost, while is the more prominent aspect in the graphical notation. Nevertheless, the use of a notation for the movements permits to capture some information about *how* the movements are performed by two users, therefore about the qualitative unfolding of mathematically thinking in movement.

Additionally, we can add to the notation qualities and "intensive" elements that characterise the movements, as well as particular actions, as turning heads, smiles or sounds that help expand our understanding of what the experience of the experiment is like.

But, in which sense is this different from a transcription of a video recording? This is an attempt to capture regularities as well as queer aspects in the experiments, and to highlight, if this is the case, a structure as well as deviations or micro-movements that would be otherwise ignored. From the point of view of the researcher, to work with a notation engenders a new manner of engaging with the analysis. This is an initial response to the need of finding ways for not losing the micro-perceptual aspects that are involved in any movement whatsoever (an aspect deeply discussed in Intermezzo: Movement).

We should finally ask in which sense this notation helps speak of the collective mathematical thinking even in relation to classroom discussions before and after the experiments, and to the evolution of different experiments during the course of one intervention. Further, we might investigate the ways in which the notation highlights structures for the movement with respect to particular tasks.

Remarkably, I do not mean the use of notation as an aid to comprehend the meaning of the theory (de Freitas, 2012), rather it is a way of engaging with data and analysis through a different tinkering method. The notation does not want to capture the essence of a movement, but aims at describing movement in detail, as generative of new ideas and as a way of bringing to the surface new questions.

5.3 Quality of results

5.3.1 Validity, reliability and generalizability

Validity concerns the extent to which a research method measures or studies that which is claimed to be measured or studied (de Freitas, Lerman, & Noelle-Parks, 2017). Reliability is about independence of the researcher (Bakker & van Eerde, 2015). These concepts are entangled with ideas of transferability, replicability and generalizability of the study, especially in the context of design-based research, for which disseminating the environment or tool under study is a crucial research outcome, and impact on a wider population than that taking part in the experiment is desirable.

In this study, I address the issue of validity by adopting a methodology of data collection that allowed capturing rich data, then by analysing these data through a variety of methodologies. My aim is not that of claiming the study generalizability, but rather, following Nemirovsky (2011), my data analysis attempts "to generate rich, evocative descriptions of lived experiences, enabling insight into someone else's (as well as one's own) subjectivity" (p. 316), within the mathematics classroom.

5.3.2 Ethics of data collection and analysis

At the beginning of each intervention, the senior researcher presented all the people involved in the study and explained the aim of the experiment as not that of evaluating the students in any way, but that our interest lay in their ways of moving, talking and thinking during the activities. At the same time, we asked the students to be responsible for their own behaviour in the classroom, for the relevance of their engagement in the research study. The teachers were selected were committed to dedicate ten curricular hours to the development of the teaching experiment. The students were informed, and their families furnished consent to participate in the intervention as well as to the video recording of classroom activities.

Throughout the data collection process, I played several roles, including observer, interviewer, and participant observer. During the days of the interventions, I managed the video recordings and the technical issues related to the software's use, when necessary. In addition, I was involved in keeping the timing of activities by signalling time interval to the researcher and checking the functioning of the video recording equipment. During the collective discussion, I could also intervene, mainly posing new questions to the students.

Another way in which I was positioned in the learning event was as a designer of the tasks. At the end of each day, the senior researcher and I met, shared general impressions about the day, and eventually re-designed and adjusted tasks for the following session. During group work, I videotaped a selected group of students and my interventions were usually restricted to general issues regarding their tone of voice or timing in the activity.

Often, the students asked me questions about the worksheets, but I avoided giving them specific suggestions and answers. Nevertheless, I always acted in that context trying to loosen up the tension of my external presence in the activity, smiling and encouraging them to proceed independently, until they became familiar with that situation.

At the end of the experiments, I was also the interviewer of some students. Even if, I managed the interviews in place of the senior researcher, who had already covered the teacher role during the intervention, I was aware of the inevitable power imbalance and the age gap between interviewer and interviewee. Before beginning a new interview, I reminded the student that I was neither going to evaluate her in any way nor expecting "right" answers. I tried to establish an informal atmosphere and to pose questions without being interrogative or totally imposing the course of the interview, rather attempting to follow the student's own reasoning and to accommodate the issues she was bringing to the fore. I believe that this was a fair way of being responsible for the other person in the room, without valuing my solely research interests.

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[1] LabanWriter is downloadable at: https://dance.osu.edu/research/dnb/laban-writer. From the website: Labanwriter is a Labanotation editor for the Macintosh developed by the Ohio State Department of Dance. It can be downloaded for free, and the current version will run on any Macintosh computer system running OS 10.4 or greater. Older versions will work with system 6.01 to 9.5. LabanWriter is a software program that permits dance to be copied, edited and stored on a computer. It utilizes the symbols for Labanotation, a movement language devised by Rudolph Laban in the 1920s, to record dance on paper. The program includes more than 700 symbols that indicate parts of the body, direction, levels, and types of movement and the durations of each action.

6

Episodes

As exploited in the previous chapter, my orientation towards research was qualitative throughout the data analysis. In the whole process, I located my approach within work from different fields while articulating and extending this work through different methodologies, with the aim to investigate the subject of study with considerable openness and a diversity of lines. The initial broad research interests, primarily in the entanglement of perceptual, diagrammatic and symbolic aspects of the mathematical activity and in the role and constitution of movement in mathematical practice, were refined and came to be more and more focussed during the process.

This chapter first presents the research questions of my line of study, in light of the theoretical and methodological commitments that I have detailed throughout the previous chapters. Then, selected episodes from the interventions will be presented and analysed. In particular, the initial episodes come from the pilot experiment, in order to discuss emerging insights into research. The episodes have been organised in subsections that are structured along several lines or specific themes. These subsections are not meant to provide the reader with a chronological order for travelling across the interventions. Rather, they aim at unfolding different themes from lower to upper grades, or at centring on main aspects as they emerged in a specific grade.

6.1 Research questions

Mathematically speaking, when modelling a situation, graphs are meaningful to describe processes. Creating or interpreting graphs that model a process means to say something about that process. When we think of motion graphs, we can say that the graphs describe the movements that produced (or can produce) those graphs. Conversely, one might ask whether the movements themselves could say something about the graphs. In other words, *if a graph is a particular semiotic notation for the bodily movements, can the movement be some kind of notation for a graph?* Engaging with this question means to investigate how the qualitative nuances of bodily movements enter the mathematical concepts of function and graphs. I explore this issue drawing on two lines, which rely on my commitment to the role of the body and movement in mathematical thinking and learning. These lines are investigated through three main research questions:

- *How is the mathematical experience of students using WiiGraph to explore spatio-temporal relationships playing out across entwinements of perceptuo-motor, symbolic/diagrammatic and imaginative aspects of the activity?*
- *Which mathematics does it emerge out of the activity? Or better: How does the mathematics change in the encounter of the students with the software?*
    - *Specifically, regarding the event of crossing lines: What kinds of meaning does this event generate in the mathematics classroom for the concept of function?*
- *How does a collective movement of thinking emerge and get distributed across the learning assemblage?*

For the sake of analysis, we need to introduce two specific terms that I will be using in the discussion of the episodes.

Configurations: drawing from physics, where a configuration space is used to describe the state of a whole system as a single point in a high-dimensional space, and from mathematics, where geometric configurations are finite sets of points, and a finite arrangement of lines, such that each point is incident to the same number of lines and each line is incident to the same number of points, I use the term configuration to speak of any arrangement of lines or hands, imagined and/or actualised through the bodily, with diagrams or in words. What is relevant in the idea of configuration are positions and relations between the elements that constitute the space or that are immersed in that space. The configuration is only one among the potential ones that those elements can create. Speaking of configurations allow us to detach from the logic of examples, while claiming a generality in the particular instance concerning the relationship between the parts that compose it. The term is only used at interpretative level, that is, with the aim of analysing the particular couples of graphs that the students produced or worked with. It was never used with the students.

Choreographies: as is with any dance, a choreography is a sequence of movements co-occurring in a shared space and involving one or more performers. Choreographies could be performed by bodies in motion, but also by hands and moving remotes, or be recalled in words by motion narratives.

6.1.1 Episodes selection

The episodes that are presented and discussed below come from the teaching experiments that have been detailed in Chapter 4. The diagrams offered in §4.5.7 should have given the reader an idea of the complexity of the interventions. The methods of data collection and analysis described in Chapter 5 have provided the most important analytical tools that will be put at work here, together with the theoretical commitments of Chapters 2 and 3. This chapter lies at the conjunction of the methodological, theoretical and epistemological dimensions. Therefore, the rationale of the episodes' selection takes into account all these dimensions.

In particular, borrowing from Derry et al. (2010), I use more than one method of representation for video analysis when reporting the episodes, with the aim of achieving compelling images of complex interactions. Briefly speaking, the main methods employed in my research are: dialogue transcripts of selected episodes, narratives about the most important happenings that precede or follow the chosen video segments, descriptions of gestures, tone, movements, and facial expressions, still images of bodily interactions, notations for movements occurred during the experiments with WiiGraph. The selection strategy for the episodes has also involved identifying principal themes that emerged from repeated viewing of video data. The themes do not implicate a systematic coding of the available data. Rather, they want to offer multiple glances at movement in (relation to) mathematical activity and mathematical thinking. They are meant to create possibilities for the virtual dimensions of mathematics and the phenomenological nature of movement to emerge by means of meaningful case studies. Therefore, each theme stresses the broad interests in movement and gives insights along the lines brought forth by the research questions. Finally, each names a specific section of the chapter, while the selected episodes are organised in subsections. Most episodes traverse both the theoretical commitments and the different classes involved in the study. As already mentioned, I have chosen to avoid the chronological order for presenting the activities, renouncing the way in which they were faced. I rather want to leave with the reader a flavour of the entanglements that come to live in the study and that emerge from the investigation of similar situations in different contexts.

In concert with the verticality of the design (detailed in §4.5.3), the analysis wants to be *vertical* too: most of the discussed issues traverse the specificity of the grades in which the interventions were carried out, and we can instead identify main common points, no matter the previous background, knowledge or age of the students.

Some of the episodes have been chosen in order to focus on segments of the different teaching experiments (§6.6, §6.7), some other according to specific aspects that cut across the different grades (§6.3, 6.5). Section §6.4 discusses episodes coming from each teaching experiment and concerning similar tasks designed for parallel activities. Sections §6.8 and §6.9 are centred on a specific modality of using WiiGraph but deal with the experience of moving from different perspectives. The chapter, in fact, closes with a first-person experiment in which I share with the reader my personal experience of thinking in movement. Within the sections, I will alternate the use of the past and the present tenses for the verbs respectively when I describe what happened and when I enter into an interpretive vision of the episodes. The following section begins with a discussion on the relevant issues emerged during the pilot experiment.

6.2 Emerging issues and first results from the pilot experiment: An overview

This section will touch on major issues, coming from the pilot experiment, which are relevant for the analysis. Specific attention will be drawn to the episodes concerned with the particular configuration of crossing lines as it emerged in the classroom from an unexpected experiment, and then to the way in which it was recalled in the interview by one student, Luca, some months after the end of the intervention. I will offer some reflection on the mathematical activity of the students by means of these initial insights.

6.2.1 Pilot experiment: Using WiiGraph within a primary school classroom

As detailed in §4.5.1, the pilot experiment was conducted in a class of 21 grade 4 students from May to June 2017. A main aim of the experiment was to understand whether we could use WiiGraph with grade 4 students, without requiring them too much effort in learning how to use the mathematical instrument. During the three 2-hour sessions, the students get acquainted to the controllers very fast, and showed no real difficulties in coordinating their movements while pointing the devices to the sensor bar. It was essential for us to understand whether the whole experiment was meant to be challenging but not too complex for the children, especially inside the classroom context. We know for example that studies with graphing motion technology had involved 9-10 years old children, but mainly in the context of individual teaching interviews (e.g., Nemirovsky, 2011). Moreover, graphing motion devices (e.g., motion detectors), which do not require remotes or other devices, have been largely and successfully used in the mathematics classroom for structuring medium-term and long-term interventions (e.g. Ferrara, Ferrari, & Savioli, 2019). The pilot experiment was aimed at testing the feasibility of the intervention at primary school. The students that have been involved in the experiment had some previous knowledge about graphical representations, mainly related to basic statistical representations of data (e.g., bar graphs and pie charts). Some insights were offered by the pilot and fostered both the design process and the research process, as we will investigate in the next subsection.

6.2.2 Crossing lines

During the first day, the students met the software for the first time and investigated the meaning of the two lines. Some students noticed since the initial experiments that the position of the two students was relevant for the creation of the graphs. In particular, some stated that "the farther they went, the higher the line went" and, conversely, that getting closer to the sensor produced a descending line as a result. Particularly interesting for developing a rich discussion on the relative positions of the graphs in the Cartesian plane and the corresponding students in the physical space of the classroom was the investigation of crossing lines.

Specifically, the students encountered straight lines crossing each other “by chance”, while they were facing the task of creating two parallel straight lines. Elisa and Stefano, a first couple of students, moved attempting to coordinate with each other and created two lines like the ones shown in Figure 6.1.

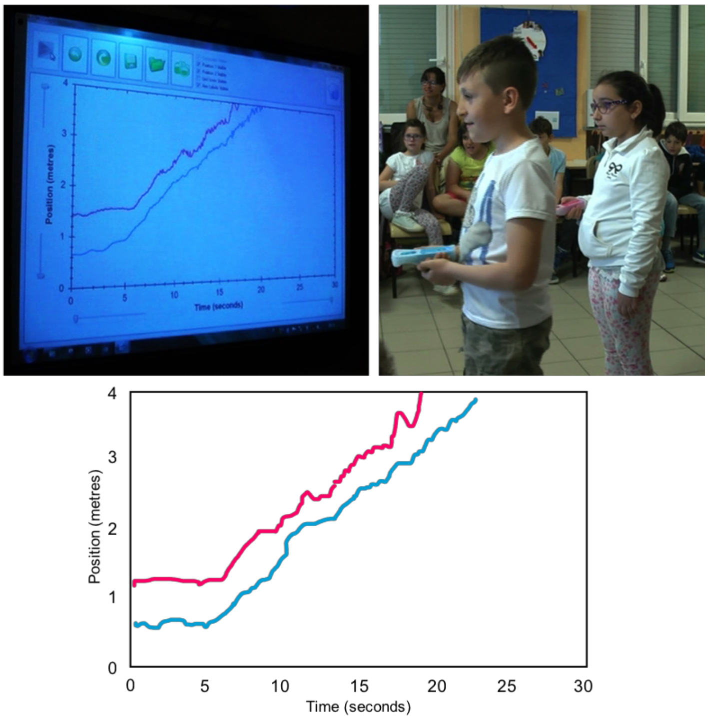


Figure 6.1. Elisa and Stefano’s experiment

The students discuss about how to eliminate the starting horizontal segments from the lines to obtain two new lines that are “just slanted”. Federico and others suggest that the two mates should not stand still at the beginning of the experiment, if they want to get rid of the horizontal segment for each line. Other two students, Giulia and Luca, intervene in different moments of the discussion by bringing to the fore an additional point: it is important to move slower during the experiment to avoid that the lines “go out” of the Cartesian plane. The researcher involves the two children in a new experiment and asks them to coordinate with each other again to produce two parallel slanted straight lines. Without saying anything to one another, the students get prepared for the experiment. For the first part of their movement, Giulia and Luca move slowly and maintain very precise coordination. They walk back at a similar speed; their pace is slowly shifting far away from the sensor; their gazes are directed towards the screen. At some point, coordination is lost: Giulia stands almost still, and Luca reaches her, so that the two lines meet, with the surprise of students who react with a laughing “Nooo!”. Luca smiles, while Giulia seems slightly worried about that particular happening, and laughs as she felt relieved only at the end of the experiment. The students immediately chorus “They met!” to the researcher’s question “What did it happen?”. The unexpected experiment creates puzzlements in the classroom, since the breaking of the requested condition (parallelism of the two lines) has evidently occurred. This makes rooms for a discussion on aspects about choreographies and configurations in the experiments as I briefly offer in the following.

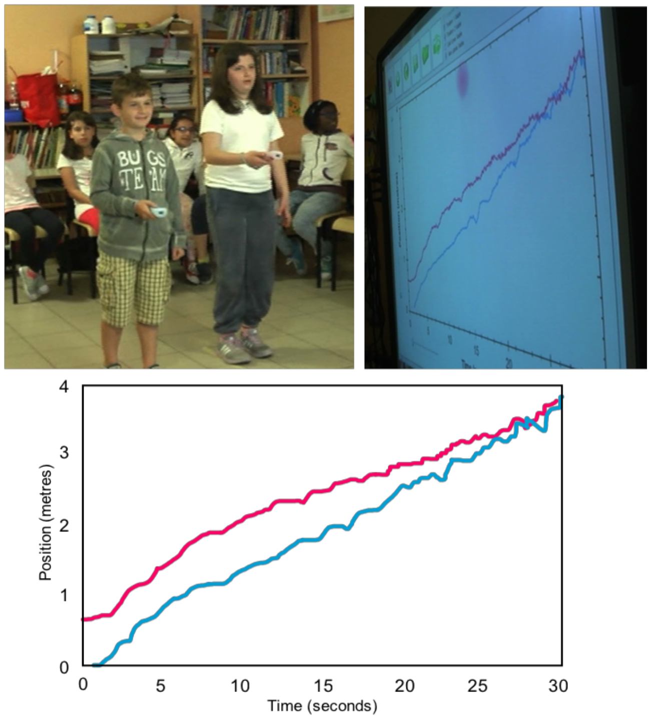
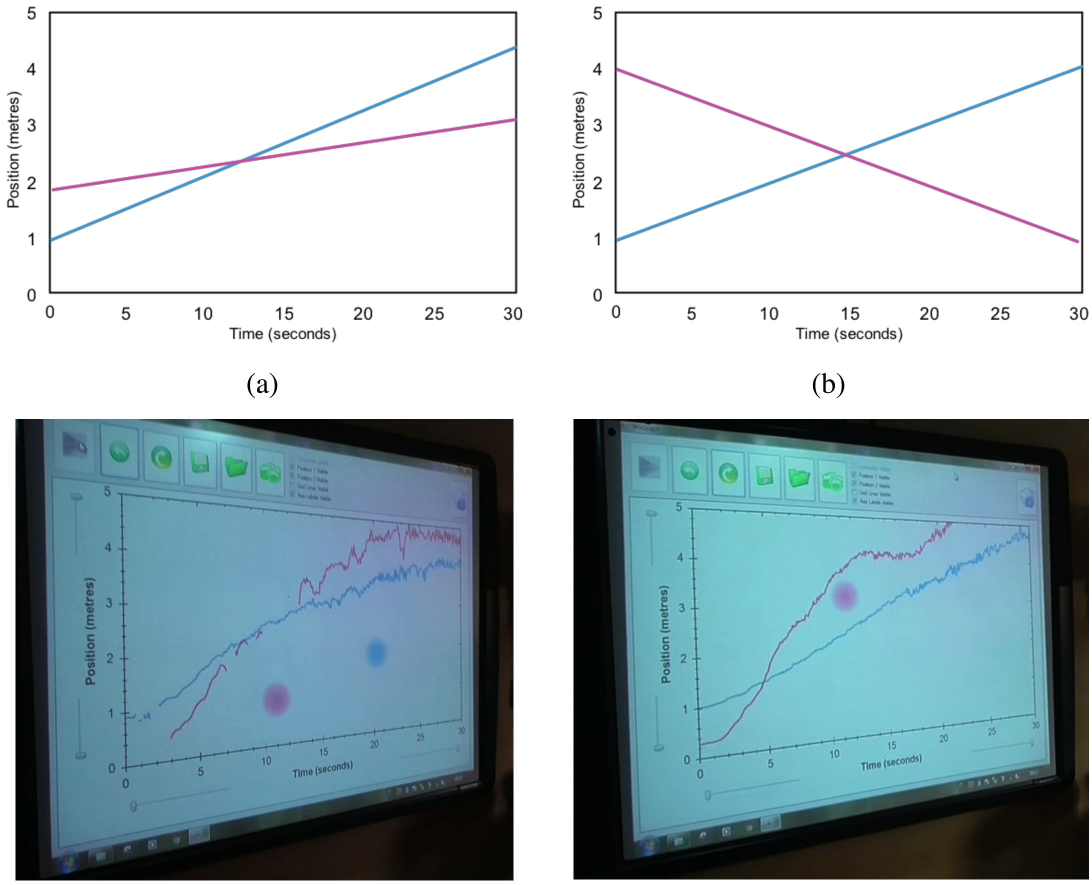


Figure 6.2. Luca and Giulia’s experiment

Going back for a moment to Luca and Giulia’s experiment, we might say that a *choreography*, in which two people start close to the sensor, one metre away from each other, and, moving at a constant speed, walk away from the sensor, is one that allows for the creation of parallel straight lines with positive slope. This is not the only choreography: there is an entire bundle of possible pairs of movements that gives rise to a similar diagram (two parallel straight lines). Of course, the constant and equal speed is a pivotal element, but it is not the only one that matters. There is a manifold of interactions between movers and nuances or little variations in movement virtually contained in that *configuration*. A similar point can be stressed if we think of crossing lines: we see four instances below. Figures 6.3a and 6.3b have been drawn with straight lines to give the reader a sense of two possible configurations; Figures 6.3c and 6.3d, instead, show each a pair of graphs produced by the students while trying to create a configuration similar to the first.



(a) (b) (c) (d)

Figure 6.3. Configurations that show crossing lines

Both examples of Figures 6.3c and 6.3d involve the crossing of the two lines and implicate that the two moving students meet correspondingly in space, but they entail different movements. In addition, referring to configurations like those of Figures 6.3a and 6.3b respectively, the students often used expressions like “to overtake the other” or “to swap places” to distinguish the ways in which eventually the students get to meet in the interaction space. We have observed variety in terms of the metaphorical language with which to convey these different choreographies and different attempts to explain the event of crossing lines, which suddenly emerged from concrete experiments in the classroom.

Therefore, we assigned a written task that involved the students in interpreting a couple of graphs like those of Figure 6.3a, and in describing which movements could have created them. Regarding my insights into the situation, the event of crossing lines, experientially encountered in the classroom through Giulia and Luca's movement, and further developed in the course of the teaching experiment, was pivotal for the students to grasp the relational nature of the graphical representations obtained with the software. While the meeting point makes the two lines collapse into something that decreases dimension (from lines to point), it also catches the generative way in which the lines separate from each other, in one or the other way (say, one overcoming the other, or simply exchanging their 'roles'), this time increasing dimension (from point to lines). Furthermore, the ambiguity generated by the expression "to meet" (lines cross but students meet) implied richness of discourse and was crucial to connect the two graphs while also making sense of the main representational system. As is for the phenomenon of *fusion* (Nemirovsky & Monk, 2000), the ambiguity arisen from the ways of speaking, without distinguishing, about the graphs and the corresponding movements that gave rise to those representations is not a confused manner of describing things that are in relation to each other, but rather makes space for the blooming of meaningful situations.

For these reasons, for the following interventions, large part of the design was centred on creating space for the students to experience and discuss about (these) kinds of configurations. At the end of the pilot, when interviewing some students, one aspect that we investigated in depth exactly concerned the event of crossing lines. We turn now to one of such interviews, which engaged Luca, the same student who partook in the experiment with Giulia mentioned above. I have chosen to investigate a 1-minute segment in which Luca and the researcher (myself) talk about different choreographies and configurations regarding crossing lines.

Luca's interview

After six months from the end of the pilot study (the students were already attending grade 5, but they were at the very beginning of the new school year), we interviewed some of the students that participated in the teaching experiment. Luca was one of the interviewees.

What follows specifically refers to his interview, in which crossing lines came again to the fore. In the first part, Luca is asked what he liked and remembers about the classroom intervention. On the table, two remotes and the sensor bar are at disposal (but the software is not in use), together with some pens and a sheet of paper. He begins telling that "two children held the remotes, and they had to do lines on that graphical area, pointing the remotes to the sensor". Then he speaks of the case of parallel slanted lines as that in which "two students had to go forward keeping the distance fixed". Holding the remotes, Luca and the researcher simulate this experiment following the indications given by Luca.

After few minutes, the interviewer askes Luca about the crossing lines:

(L = Luca, R = researcher; L/RH = left/right hand)

1.	R:	What if I wanted to create two-o lines that cross each other, at some point?
2.	L:	It is necessary that a child goes forward (*RH, holding a pen, moves rapidly towards his torso, then comes back to the starting position, in front of him*), th-, the other goes faster (*LH goes shortly back and then with impulse reaches RH*) and then they have to meet (*slowing down speed, LH reaches RH. Looks at R*) (pause) in a point (*still gazes at R*)
3.	R:	Can we try out? What would you do? (*takes one remote in RH and keeps it pointed to the sensor in front of her*)
4.	L:	(*takes the other remote with RH, gazes at R's remote*) I start ahead, then you go faster (*moves LH index finger from the R's remote position towards the sensor*), I go slowly and then they meet (*LH reaches his remote, fingers extended and kept in the same position for few seconds: Figure 6.4a*)
5.	R:	And do we both move forward? (*LH rapidly points to the sensor*)
6.	L:	No, then they meet (*LH goes back, then slowly goes forth again and overtakes his RH*), then you go forward, and I stay behind (*RH zigzags moving a little closer to his LH*)
7.	R:	Ok
8.	L:	So, you do, (*LH points to the sensor*) you go ahead
9.	R:	Tell me when to go (*keeps the remote still*)
10.	L:	Go (*gazes at R's remote. R and L move the remotes towards the sensor*). You do like this (*moves his remote a little back*), you overtake me and I stay behind (*looks at R, moves again his remote towards the sensor*)
11.	R:	Ok (*interrupts her movement*). So, how are the lines showing up?
12.	L:	(*puts the remote on the table*) Criss-crossed (*cross arms*: *Figure 6.4b*)
13.	R:	How?
14.	L:	Hm (*cross arms again, turned to a different slope*, *takes a pen*), do I draw them…? (*softly speaking*)
15.	R:	Yes, yes, as you want (*puts her remote on the table*)

16. L: One like this (*draws line 1; Figure 6.5*), the other one like this (*draws line 2 in Figure 6.5*)

17. R: (pause) (*gazes at the drawing*) So-o (*points to the drawing*)

18. L: Hm, no (*with closed fists, RH ahead and LH back are swapped in position*), I start ahead and then (pause) I start ahead (*points ahead with RH*), you start from behind (*points back with LH*) and then (*suddenly, swaps hands' positions again*) they cross each other

19. R: So, is this drawing (*points to it again*) of another movement, for you?

20. L: Yes (*takes the remote*)

21. R: So (*takes the remote too*)

22. L: I start ahead, you [start] from behind, they cross, then (*Figure 6.4c captures the experiment performed by Luca and the researcher*)

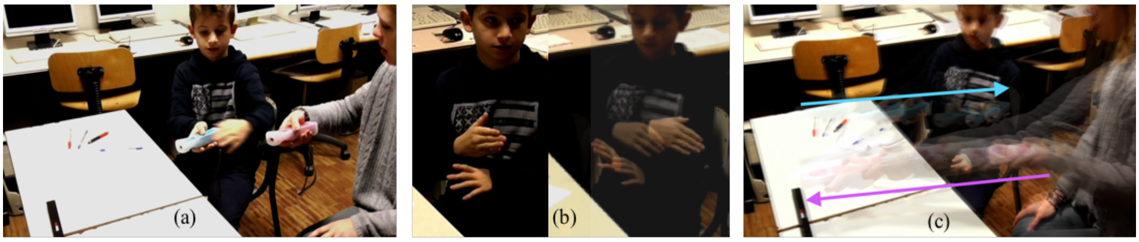


Figure 6.4. (a) first choreography; (b) gesture for intersecting lines; (c) second experiment

We see how, in the interview, two different mathematical events mainly resonate with the question and the experience of crossing lines (Figures 6.4b and 6.4c). These events emerge as intertwined out of movement and fuel Luca's thinking in movement. Bodily movements actualise specific choreographies, perform simulations of experiments, establish shapes for diagrams and arrangements of lines (configurations), or even mix the three aspects together. We can capture a sequence from the episode. A first choreography sees hands, people, remotes going in the same direction, then meeting and eventually one overtaking the other (three times: [2], [4], [6]; Figure 6.4a). Then, a first experiment also engages the researcher in performing such choreography: [10]. The following configuration (two gestures, one diagram: [12], [14], [16]; Figures 6.4b and 6.5) with the emerging diagram is a turning point as it reconfigures previous movements and engenders a second choreography. The new choreography (2 times: [18]) is still evoking the crossing relational movement, but now involves two hands, people, remotes swapping positions. Finally, a second experiment, rhythmically dictated by Luca's narrative, closes the episode ([22]; Figure 6.4c).

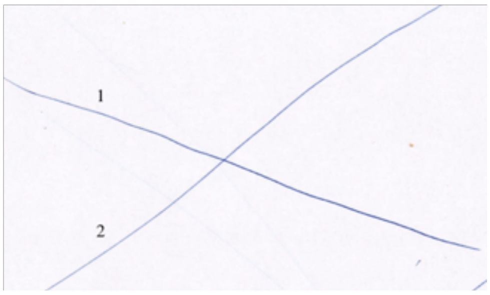


Figure 6.5. Luca's diagram: 1 and 2 indicate the order in drawing the lines

Each moment fluently evolves into the next in the experience of thinking in movement, which I characterise as follows. On the one side, the diagram reconfigures boundaries between the two choreographies by unfolding a new point of view that is also able to capture a crossing event. Hesitation and suddenness destabilise homogeneous continuity in the temporal overlapping of the two possibilities, as well as of the processes of thinking and moving. On the other side, repetitions of a choreography entail little variations within movements, as is in the case of the first choreography, where a zigzagging of the remote is added in a way that stresses relative positions between hand and remote, and therefore in the two movements. *This sheds light on the complexity and profundity of the process of movement in thinking and the potential dimensions of both moving and thinking*.

Ambiguity between the choreographies is generative of new meanings that are still open to mobility within the mathematical event, which is at the core of the episode. We see how the choreographies that are at play in this event mesh together two different events, in which the most important aspect is not the reference system, but is the relationship between movements, between hands, between remotes. In a sense, one might be tempted to assert that Luca does not remember what far from, or close to, the sensor means for the graphs. But rather, this analysis wants to bring forth that in this particular context of modelling motions what does really matters in the first place is how the contemporaneity of movements in space generates couple of graphs that relationally attuned to one another by means of potential variations within movements. Such mobility and openness resonate with the creative power of explosion attributed from Leibniz to points, when thought of as generated by the intersection of two lines or curves, as discussed in Châtelet's account of the virtuality of mathematical concepts (see Chapter 2, §2.2). In this perspective, we interpret the crossing of lines as generative in the context of explorations with WiiGraph.

Moreover, if we turn attention to the different intensities that populate the interactions between Luca and the researcher, we can grasp how the relationship between graphs is a modulated effort of coordination and little movements. The movements occur more or less on a linear path, but are subjected to zigzagging [6], impulses [2], slowing down [2; 6] variations, which emphasize the relationship between movements (the constraint of being behind the other, faster or slower speed). Then, the sudden change, after drawing the two lines, overlaps with the previous choreographies, bringing forth a new one, which privileges relative positions in space ("I start ahead and then… I start ahead, you start from behind"). Luca first actualises this new choreography in gestures, with initial slow movements that, through repetition, become quick and abrupt. Then, the researcher tries to accommodate the rhythm dictated by Luca in words, which also accentuates a third moment, namely that of crossing each other (possibly the lines, the remote, or even both). Focus on these little variations creates the possibility of appreciating how thinking is intermeshed with the nuances that are created in movement.

In the episode I just presented, we see how Luca and the researcher are exploring meanings for lines that cross each other in WiiGraph. In imagining movement, he is materially re-creating the (interaction) space and causally relating this movement to the intersecting lines that the researcher has only discursively evoked. The immersion in the Cartesian plane is lived and experienced simultaneously in terms of relative motions (persons/controllers) but also in terms of moving lines that interact with generative (not only dialectical) relationship the one with the other. Gestures that capture relative motions are here simultaneously a way of knowing and a creative mathematical act.

In the next sections, I will again touch on the explorations of crossing lines and investigate how these can be considered pivotal in thinking of graphical couples of graphs with WiiGraph. Classroom discussion around this point created new meanings for the intersection as "swapping places" or "overtaking the other", which are the configurations captured by the choreographies in the segment of Luca's interview. As researchers, this point made clear for us the importance for students of experiencing and making sense of the intersection of lines in order to relate not simply each of the graphs to an individual movement, but the graphs themselves, as well as the movements, to each other. Examining movement in thinking in Luca's interview, I offer a way of drawing attention to how the flow of movement implicates dynamic thinking about pairs of graphs and their relations, being generative of mathematical meanings beyond its own meaningfulness. I have even used superposition of subsequent video frames with increasing transparency filter (see again Figures 6.4b and 6.4c) to induce a sense of movement which cannot be otherwise grasped by still images (a delicate methodological issue already discussed in Chapter 5, that is the search for suitable methods, which allow for better addressing and capturing the complexity of movement without reducing it). This also points out the richness and hidden beauty that emerge from the challenging matter of movement in/of mathematical concepts, which may be infinite source of delight or, as Châtelet would say, "enchant(e)ment".

6.3 Exploratory Experiments with *Line*

This section examines some of the exploratory experiments with WiiGraph that occurred in the first days of the different classroom-based interventions. The episodes capture the students' first encounters with the software, the very first interactions and the graphical configurations that emerged in the initial phase of each intervention, during the collective discussion led by the senior researcher. These experiments are also crucial for the students' formulation of the first conjectures about *Line* graphs and constitute a moment in which the freedom of movement manifests itself in the mathematics classroom, with different styles and conditioning differently the next activities of the teaching experiments.

During the pilot experiment, the first exploration with WiiGraph in grade 4 was particularly striking me: the two children started jumping back and forth, laughing and provoking the laughs of all their classmates. One of them started first, the other quickly mimicked him in the repetitive movement, to the point that the researcher asked them to not follow each other and to stop jumping (since that prevented them from correctly pointing to the sensor). Even if the researcher intervened to avoid that the experiment would have 'crashed', the students seemed to enjoy that first exploration so much. The incident actually brought forth some elements, which I consider crucial in all the exploratory experiments. I list them in few words. First, this exploration highlights the immense power and freedom of movement, which manifests itself in unexpected ways, like the movement of jumping back and forth. A second point is the mutual influence that is established between the two children in their movement, as they move next to each other and somehow give the impression of following one another. As a third issue, the collective engagement of the classmates that are partaking in the experiment with laughs, directions and suggestions, while observing the experiment, extends the individual and pairs' perception to a broad sense of *affective attunement* to the exploratory event. This section aims at characterising the qualities of these experiences, which are very diverse from one another in their overall nature.

Before each of the experiments we present in the following subsections, the researcher gives the students a little preamble. She introduces the controllers (that are usually well-known devices among the students and mostly create excitement and expectations in the class) and discloses that they will be used with a software application, which allows for movement experiments (WiiGraph); then, she asks for two students to come to the centre of the interaction space and perform the first experiment. The only instructions she provides them and the whole class are that the remotes have to be pointed to the sensor, and that the remote is correctly pointed whether one can see a dot on the screen of the same colour of the remote (blue or pink). Then, the researcher asks the couple of students to move freely in a corridor-shaped space, facing the sensor and the screen, and trying to maintain the controllers correctly pointed.

Additionally, at primary school, we put a 4-meter long tape strip on the floor as a reference, so that the children could move each on one side of the strip. Below, we begin from a discussion on the first exploratory experiment at primary school.

6.3.1 Primary school: Grade 4

First experiment: Thao and Bianca

At the beginning of the first day, the senior researcher wonders whether the class already know the controllers: everybody apparently does, as they raise with enthusiasm their hands to volunteer for the first experiment; everybody except for Thao and Bianca. Differently from their classmate, the two children explain that they never used the controllers (to play games), therefore the researcher invites them to be the first to use the software. Thao is reluctant but agrees and Bianca also timidly accepts.

After few indications from the researcher, the experiment starts: Figure 6.6 shows the students in the interaction space and the graphs (both from the video recording and the software's capture. I have to observe that as the students moved, the labels for the two axes were hidden, so that they could not read them on the screen, even though here labels are added in the capture for the sake of clarity. We can also see that the experiment lasts 20 seconds).

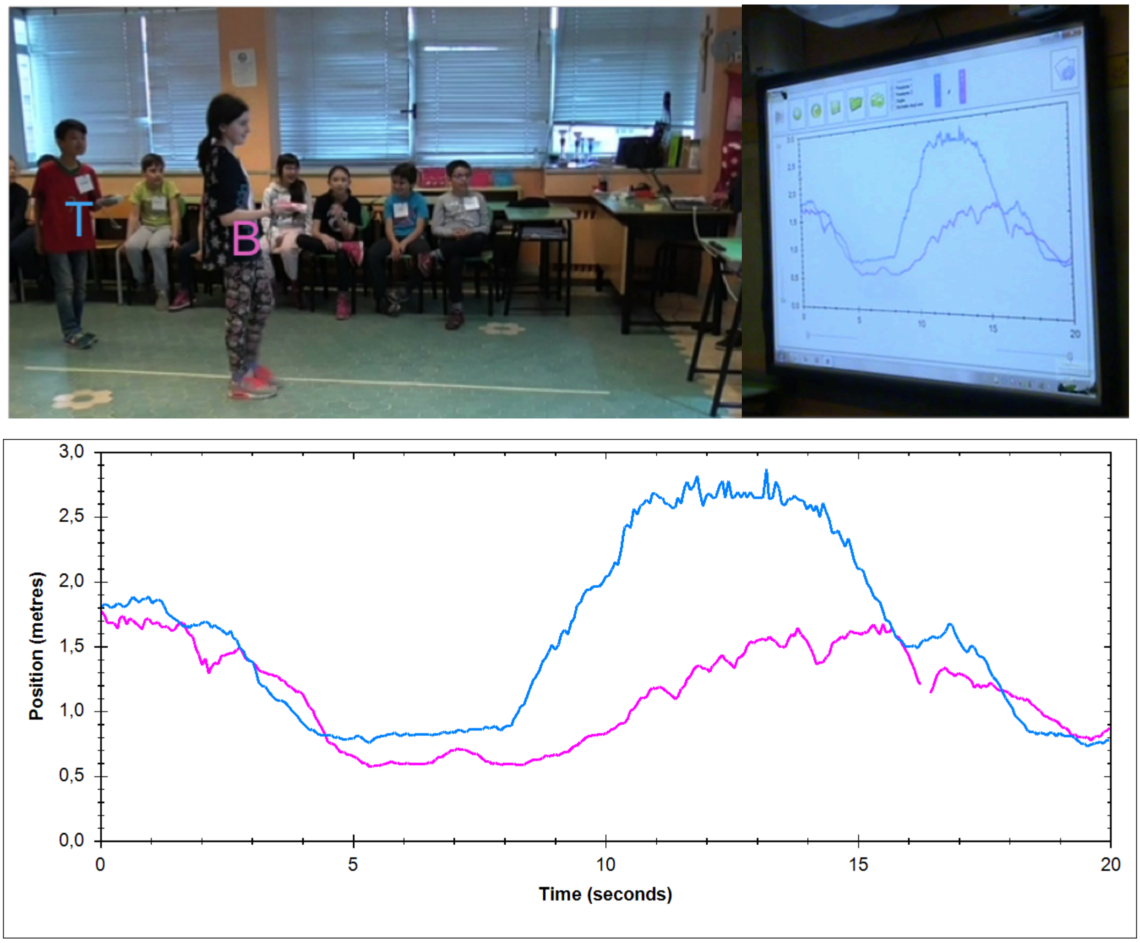


Figure 6.6. Bianca and Thao's exploratory experiment at primary school and their graphs

Figure 6.7 shows the movement notation for Thao and Bianca's exploratory experiment. In this experiment we see Thao and Bianca approaching the exploration with WiiGraph with considerable shyness. Initially, they barely move and remain next to each other while slowly shifting forward. When Thao asks: "Do we have to move?" (around 6 seconds from the start), the students are close to the sensor and almost stand still in the same position (at the same distance from the sensor, which is located on a table on the right side of the images from the classroom in Figure 6.7); Bianca suddenly giggles.

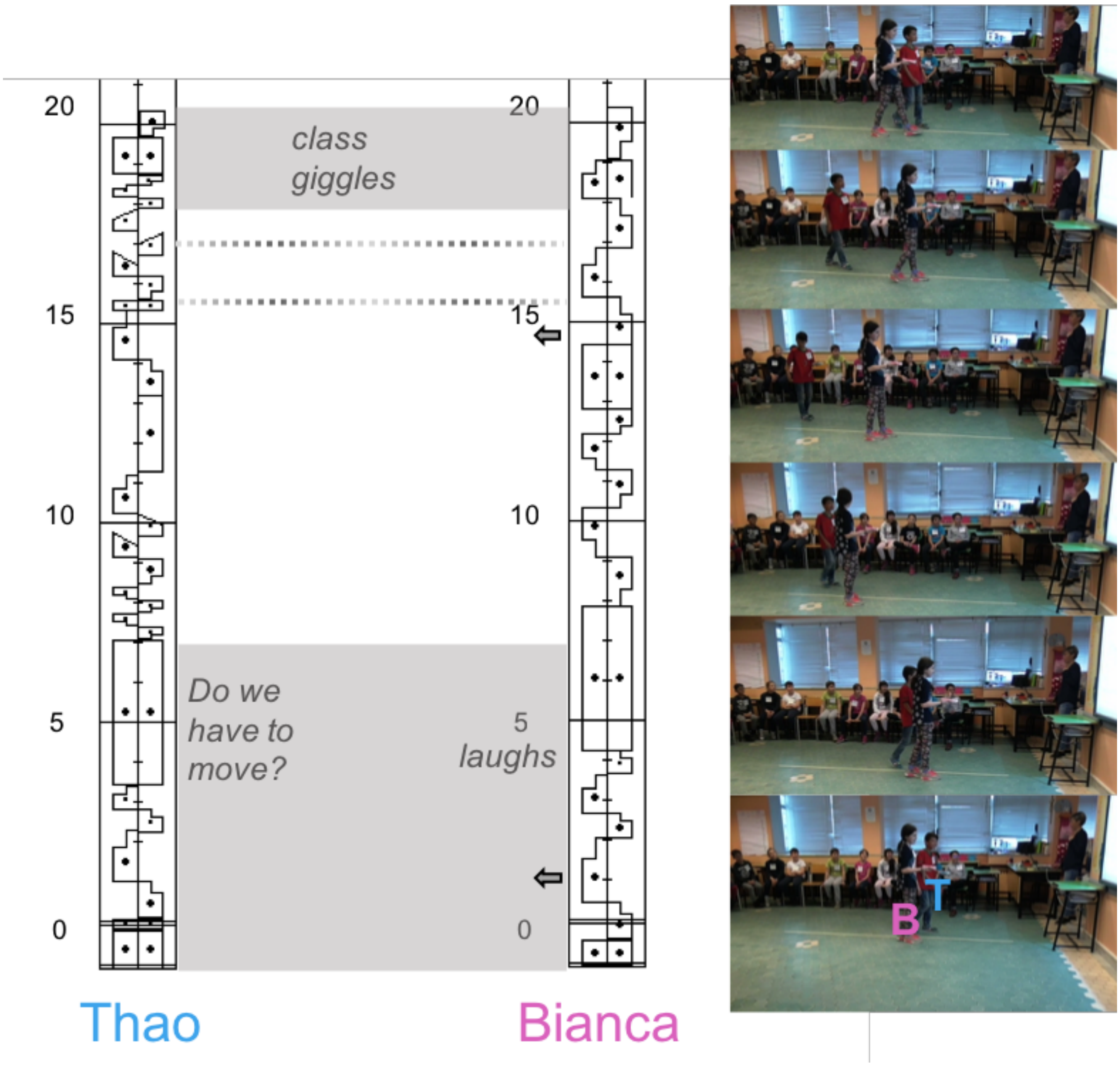


Figure 6.7. Movement notation for Bianca and Thao's experiment

As the researcher utters "Yes, yes, you do have to move over there", Thao suddenly mumbles "Ah, ok, ok" and as rapidly he changes his movement and shrugs repeatedly: he begins shifting backwards with more impulse, but he quickly reaches the end of the strip and reduces his pace. With some delay, Bianca also begins walking backwards and, gazing to her left, looks at Thao who is reaching her again. Thao moves back and forth with little steps, almost remaining in the same position, and laughs; a sort of puzzlement for this queer movement by the rest of the children move them to collective giggles for the last few seconds of the experiment, while the lines seem to interlace one another.

These descriptive elements are also captured in the movement notation (Figure 6.7), which helps us see how the two students, without being explicitly or overtly coordinated with one another, are influenced by the presence of others in moving with WiiGraph.

Thao and Bianca affect each other and look for each other during their movement, especially Bianca (see the arrows inside the notation, which signals her gazing to Thao), and they remain close to each other for great part of the experiment (grey shaded area in-between the staffs). The researcher's utterance, in response to Thao's question, fuels the abrupt movement of the child, who seems to move just in reaction to that particular prompt. At the beginning of the experiment, the children sitting all around are silently watching; the experiment closes with their sonorous giggles and laughs, which saturate the auditory dimension of the experience.

Everything happens quite rapidly and, on a first view, with lots of changes and movements occurring simultaneously. The notation aims at grasping the petite movements that lead to the graphical representations shown in Figure 6.6. Paying attention to movement through the notation and to its emerging qualities, which we highlighted in this brief episode, sheds light on the potential of considering the graphs more than a simple representation for the children movements. Rather, the graphs are populated with tensions and uncertainties, and the overall experience of exploring through the use of WiiGraph as well. The graphs are a collective production of affective forces, which sustain the movement of the two students at the centre of the classroom. Therefore, we can see how the bodies are at the centre while rethinking the way that, at the same time, they are not. The bodies are not simply in space, but rather create space (and space of meanings for their graphs) through the movements along the tape strip and their qualitative variations: sometimes hesitant, then sudden, then again with reduced intensity.

We can also interpret the whole graphical production (i.e., the juxtaposition of the two lines in the same Cartesian plane, neither just one or the other) and the relationships or similarities between the two graphs as the complex entanglement of all these forces, the tendency to look for the other, or the embarrassment created by the situation, or even the abrupt response to the leading figure's utterance (the researcher's prompt).

From the collective discussion

After the experiment, the students who were sitting engage with a discussion led by the researcher. I track here some of the utterances of the students who did not partake in the experiment but do intervene in the discussion. These interventions bring to the fore some important aspects about the first encounter with the software and the first hypotheses that were then investigated further through new experiments. The researcher asks: "Can you tell me what do we see? What do you see? What have you observed?". Andrea claims: "A graph!". The researcher invites Alessandra, who raised her hand, to intervene: "The remotes were connected to the computer and when they moved (*back and forth gesture towards the tape strip on the floor*) the lines they (*draws a doodle with the index finger towards the IWB*)… for example, Thao went further [than Bianca] and the line went higher…" Giulia intervenes shortly after and suggests: "To me, it seems like the graph of when someone moves […] it looks like a graphs about when people move". Triggered by the researcher's questions about her hypothesis, she adds that "a graph shows something" and "in this case [it] shows when people… when Thao and Bianca moved […] that is, Thao moved, he went backwards and the bar [*the line*], the blue bar went up". Another child, Martina, added a qualitative observation to this causal relationship: "To me, it seems like Thao moved a little faster and a little more behind [her], and Bianca less fast and I saw that his line is also a little higher than Bianca's". It is only after this that Dafne reformulates previous conjectures adding that: "That, if you moved forward… closer to the bar… hm… the lines went down, instead if you moved backwards, always along the strip, the lines went up".

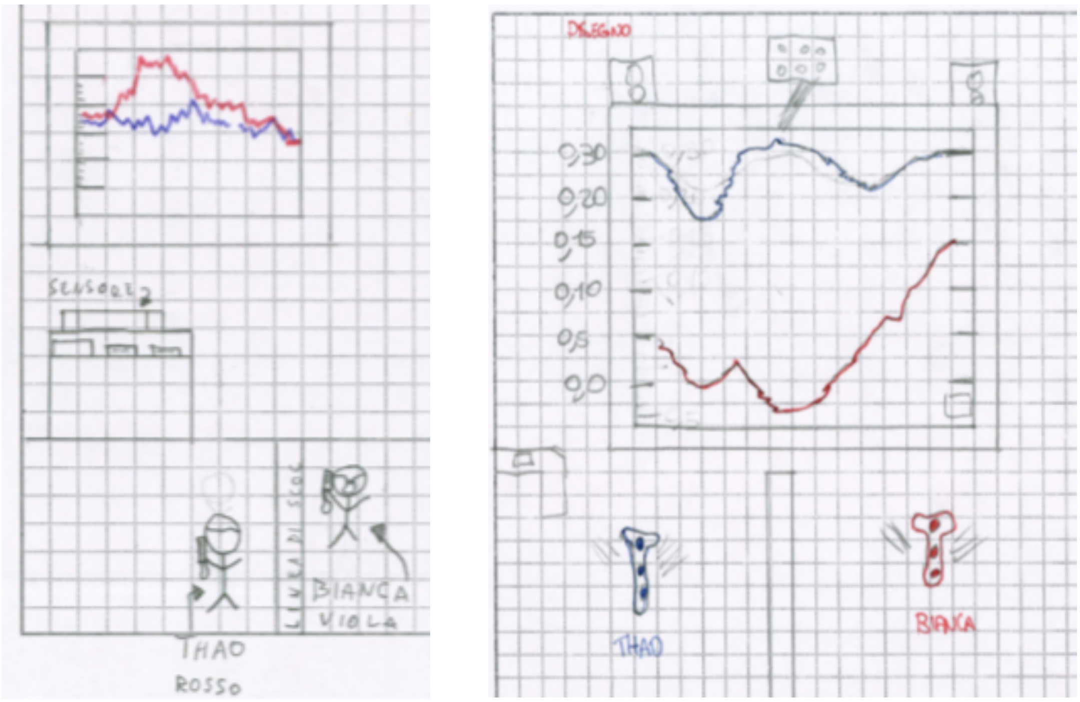
Analysing Bianca and Thao's experiment with a focus on their bodily interactions, we can say more than simply that the students were moving in space to exploit the software with back and forth movements. The analysis supported by the use of the movement notation aimed to capture the relationality of the students' movements and the complexity from which their bodies emerge in movement. This also opens up a discussion of how this experiment is a collective production as the researcher's interventions and the classmates' laughs assemble with, and affect, the students' individual ways of inhabiting the experience. This collective effort manifests itself in words after the motion experience, that is, during the collective discussion in which the students, who previously were not at the centre, could bring to the forefront their own experience. Alessandra, for example, by comparing Bianca and Thao's movements, focuses on the emerging relationality of the lines ("further"; "upper"). Giulia tries to formulate a definition for the lines, by arguing that they are the graphs for "when people move", and she recovers the fragment of the experience in which Thao's evident backwards movement makes the blue line moving “higher” towards the edge of the IWB. Martina adds a qualitative nuance to the experience, as she says that Thao moved “a little faster and a little more behind”. Going on, Dafne reformulates Giulia’s initial hypothesis taking into account only the relationships between a single movement and the corresponding line but subsuming both the forward and backwards movement with the remotes.

The students continue the discussion for almost one hour and investigate and negotiate further their observations and arguments. Nevertheless, from this brief initial discussion, we can begin tracing the ways in which the couples of movements are interlaced with the graphical productions from another perspective: that is, the perspective of the students who observed the first experience with WiiGraph. The students’ interventions I proposed here give a sense of how WiiGraph entails perceiving the graphs as a couple, therefore in relational terms. I understand these interventions as entailing an ontological move onto the event of exploration: one movement (one graph) exists *since* another one (movement or graph) exists. The relationship between the single graph and the corresponding movement is not erased but emerges out of the degrees of difference between the qualitative natures of each element in this relationship. Also, one might argue that the difference is even more apparent since Bianca and Thao stayed quite close to each other for most of the experience, and Thao’s abrupt movement backwards visually appears with significant force, while qualitatively emerging from a sudden change of direction, intension and speed. Once more, this brings forth the way in which the graphs emerge as a collective event, which enlarges and grows through multiple bodies.

From the written worksheet

At the very end of the first day, the students worked in pairs on a written worksheet, Scheda 1 (see Appendix B). They were asked to imagine explaining to a friend, who did not use WiiGraph, what they learned about mathematics through the experiments in the classroom. The students mainly described the functioning of the software by naming the remotes and connecting them to the graphs projected by the IWB and recovered various aspects that emerged during the first day, like the experience in which they explored the crossing of lines (see §6.5.1 for a discussion) and the case of horizontal straight lines (see §6.4.1 for a wider contextualisation). We can also grasp the relevance of the moment described in this subsection by noticing that some children depicted or referred to this experiment to answer the worksheet. For example, Matteo and Riccardo write: "We have understood that, when we go closer to the sensor in the classroom, a remote connected to the sensor, a line on the IWB goes down, instead, when we walk away the line went up, instead when we were standing still the line was going horizontally". In the same page they add two drawings, one of which is shown in Figure 6.8 and represents with enough precision the lines of the initial experiment of Bianca and Thao (see again Figure 6.6). In the following page, they add: "Every movement with the remote corresponded to a piece of line we drew in the previous page", stressing the correspondence between movement and the depicted lines.

William and Andrea choose to create on the second page of the worksheet a drawing that still relates to the first experiment. Their representation is less accurate regarding the lines' overall shape, but we can notice the many details added to depict the classroom setting and the number scale on the vertical axis. The two students drew the interaction space as if it was seen from a top view, while the screen is taken as if they were facing it.



(a) (b)

Figure 6.8. (a) Matteo and Riccardo's drawing for Bianca and Thao's experiment; (b) William and Andrea's drawing capturing elements of the same experiment

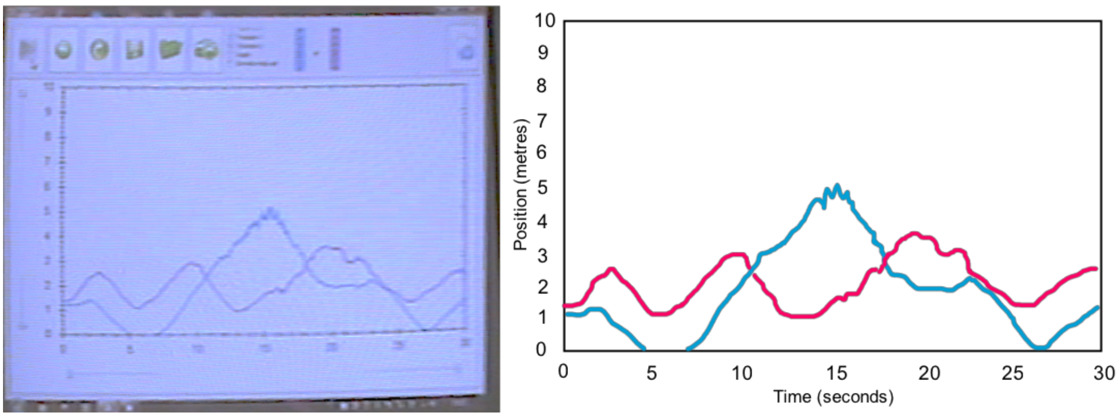
6.3.2 Lower secondary school: Grade 7

Second experiment: Vittoria and Sofia

The exploratory experiment that we take into account in this subsection was performed by Vittoria and Sofia and, differently from the previous episode, is not the first one but the second that occurred at lower secondary school, nearly at the beginning of the first day. The two students are asked to move freely in front of the sensor bar in order to further explore the use of the software. Until that moment, the entire class has just observed and discussed the first experiment, by Michelle and Alessandro. During the following discussion, the students began making conjectures related to the lines on the screen: among others, Riccardo has stated that they saw "two lines that represent the graph", and Marco has added that "the farther they went, the higher the line was".

After this, the lines are erased from the screen accidentally (someone pressed a button on the remote), therefore the researcher asks for a new experiment to be made to investigate to a greater extent these initial investigations. This is the point at which Sofia and Vittoria come to move and their experiment takes place.

Figure 6.10 shows the notation for this new experiment. Each staff captures a sequence of movements in space, for Sofia and Vittoria respectively. Their graphs are those shown in Figure 6.9 (Sofia: blue line; Vittoria: pink line). Given the initial positions of the two girls, the movement notation captures the movements that they performed in this exploratory experiment (Figure 6.10 again).



(a) (b)

Figure 6.9. (a) Vittoria and Sofia's original graphs; (b) reworked version of their graphs

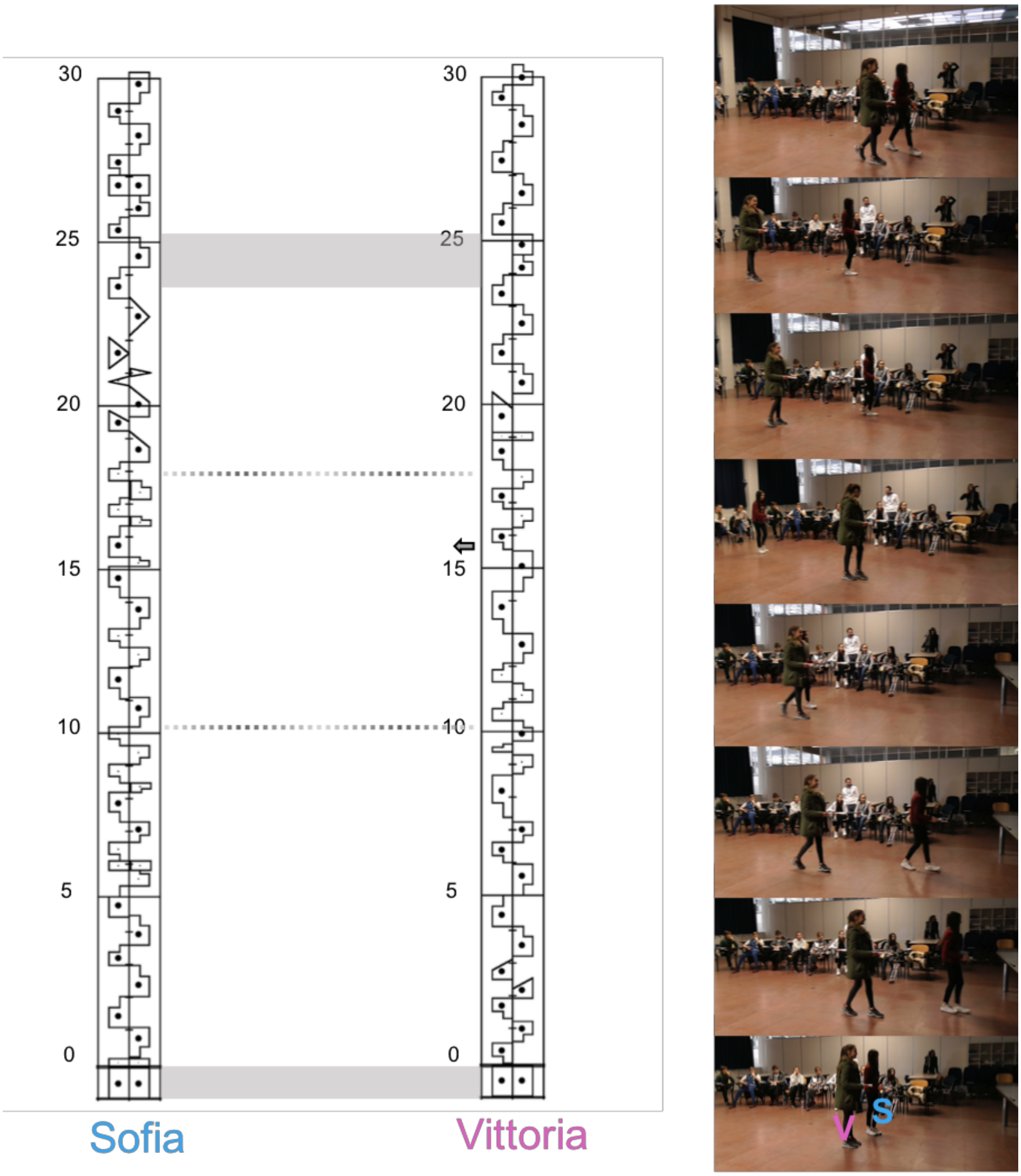


Figure 6.10. Movement notation for Sofia and Vittoria's experiment

Some descriptive elements can be derived from both the graphs and the movement notation. For example, the two girls do not start moving at the same time: Sofia seems to be slightly hesitating in the first place and stands still for few seconds. In addition, in two different moments (seconds 3-6, 27-30) the two girls move quasi-parallel to each other (keeping the same direction, moving at nearly the same pace with respect to each other), as if they were not (necessarily) coordinating with each other but – maybe – influenced each by the presence of the other in space.

These elements can be seen as common to both the graphical and notational description of movement, that is, we can see congruency between the two different ways of capturing movements and interpret them coherently. In particular, the graphical representation makes visible on the screen the relative positions of the two students in time, and the distance from the sensor over time. The relative position of the two girls is added in-between the two staffs that compose the notation, with connecting lines that signal that the students are at about the same distance from the sensor. More precisely, the dotted lines and the grey areas in the notation indicate the moments in which, during the experiment, Sofia and Vittoria cross each other and stay close to each other. The reference from the sensor bar, instead, is completely lost inside the movement notation. What we can grasp, though, from the notation – even at a first glance – are the multiple variations inside movements, which characterise the motion experience for each of the two girls. This can be spotted by looking at the very diverse signs that populate each staff (in terms of directions), and their difference in length (which mirrors a very diverse duration for each step).

In addition, we can appreciate some aspects of the girls' movements that cannot be ascribed by simply looking at the graphical representation alone, that is, that cannot be recovered entirely just looking at the final graphs as they are produced at the end of the experience. For example, we can observe the presence of a horizontal trait inside Sofia's line (corresponding to the time interval 19-22 seconds approximately). It does not correspond to the girl's stop, rather it represents when she does move with resolute steps on the left side of the interaction space, keeping the remote almost in the same position (i.e., not changing significantly its distance from the sensor), and then quickly comes back to the centre. What apparently looks inside the graph as a static and unimportant variation is instead a very important explorative move for Sofia.

We can also "see" how Sofia moves when her line disappears from the screen: she goes too close to the sensor, overtaking the origin point (which is at nearly 30 cm from the sensor). Hence, the software cannot detect her distance and her line seems to disappear under the horizontal axis. At this point, she first moves forward a little, then stops before she moves backwards again, making her graph reappear on the IWB.

On the top of the staffs some symbols go out from the 30-second limit, meaning that the girls continue to move after the 30 seconds of the given range, and they actually stop only when the researcher tell them that the experiment is over. It is as if there was no immediate feedback from the software, as the lines reaching the end of the graphical space do not prevent the girls to continue exploring. The experience of moving exceeds, with its gaps and flows, the interrupted continuity of the moving line on the screen.

Again, in this episode, the notation forces us to consider Sofia and Vittoria's exploration as more than a mere going back and forth of the students, but rather as a merging of different and distinctive intensities and relationalities. By drawing attention to the micro-movements and the qualitative nature of changes and variations, the use of a notation in this case mediates the distance between the video data and the written elaboration of the classroom experience and, more generally, between the movement and the graph. It also allows for bridging a vision of the graph as an expressive form of the two movements. This particular episode sheds light on another way of exploring the graphs' production with WiiGraph in a classroom situation. Here I have focussed less on the collective dimension of the experience, and more on the methodological gaining of using a notation. Vittoria and Sofia's interactions allow us to see that exploring also might mean going beyond the limits, trying unsafe terrains or diverging from the other. In this sense, the influence that the students have on each other when they are in movement are not necessarily leading to their identification, but instead can be productive of deviations and difference.

From the collective discussion

After the experiment, the researcher recovers the conjectures that the students had put forward: "Is it true that… […] if a remote goes far from the sensor, the line goes up?". The class murmured a "yes", while Vittoria, who stands in front of the class, says: "not always" and then adds "that it also depends on the other person".

Riccardo, from his seated position, instead, argues: "To me, no, because when… it doesn't depend on the same person because the blue line did correspond to Sofia, and the line, the purple line did correspond to Vittoria. Then, when Sofia was, when Sofia was close, she [Vittoria] didn't necessarily move close too". The researcher then invites Riccardo to the IWB to show an example of what he is talking about. He proposes: "For example, here (Figure 6.11a), when Vittoria was very low, was (Figure 6.11b) very close to the bar (Figure 6.11c) and instead, instead Sofia was very high (Figure 6.11d), was very far (Figure 6.11d). It's not because Vittoria was low that Sofia was also low."

(a) (b) (c) (d) (e)

Figure 6.11. Riccardo compares positions and lines at the IWB

Vittoria seems to be convinced by Riccardo's explanation and timidly agrees with him. The students then investigate two other issues starting from Sofia and Vittoria's graphs. First, they focus on a small portion of Vittoria's graph (highlighted in Figure 6.12), which seems to capture Sofia's attention.

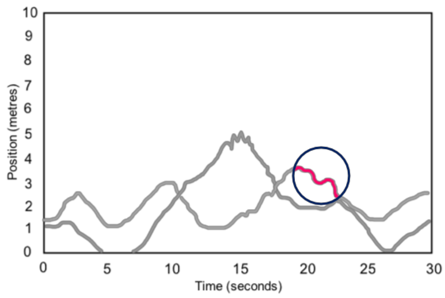


Figure 6.12. The piece of line investigated by Sofia

Sofia in fact comes to the IWB and expresses her desire to understand the meaning of that particular portion, which contains both increasing and decreasing tiny parts of the line (Figure 6.13):

"I would like to realise how she did this	[…] how she did to do this first above,	then right below	then right above"

(a) (b) (c) (d)

Figure 6.13. Sofia's gesturing at the IWB

Then she adds: "In my opinion, when she went back, that the little line was raising, that is, she went backwards (*steps backwards facing the IWB*), then she went forward again (*steps forward*), she… stopped (*hesitating*) and then she went backwards again to make it go up again, the little line".

At this point, Gianluca shifts attention to the importance of the hands' movement beside that of the whole body: "For me the remote, rather the sensor, for me, also slightly captures the movement of the remote (Figure 6.14a), because anyway, that is, when Vittoria went backwards (Figure 6.14b) it is not that she went backwards doing like this (Figure 6.14c), so that the bar [the line] did like this (Figure 6.14d), otherwise they would be straight [smooth] lines (Figure 6.14e) and, therefore moving doing like this, maybe also moving the remotes […] That is, going up and down, besides capturing how much the person goes backwards, it also captures the hand movement of the person with the remote".

(a) (b) (c) (d) (e)

Figure 6.14. Gianluca's gesturing about jagged lines

Michelle reformulates this vision slightly, by saying that: "For me, it did that little wave because she was maybe there, more or less, and, without moving, she did like this (*resolutely shifts her right hand forward*) with the remote. Maybe, if before she had it [the controller] here (*shifts her right hand back to her torso*), without moving she pulled it forward and then she went back, therefore by pulling it forward the little wave was formed down, and then moving backwards with the body it went up".

After this, other students intervene in the discussion by raising other issues. Alberto, for example, desperately wants to investigate whether the speed matters or not, and which effects speed has on the lines, and new experiments are carried out, as I will discuss in §6.5.2.

Discussion

At the beginning of the discussion that follows the exploratory experiment we see that Vittoria, one of the girls who took part in the experiment, proposed that the two lines depend on one another, or better said, that one line also depends on the other person's movement. It is difficult to argue about the reason Vittoria thinks of one line as depending on the other, especially given that it seems apparent to the classmates that Sofia and Vittoria moved in very different ways, along different choreographies of movement, and the lines described different paths on the screen. I want though to put forth a first hypothesis, which relies on the perspective on movement I am taking. It is not that movement occurs in space, it rather creates the space that the body kinaesthetically occupies. The girls are not alone, they share and modulate a common space, which I have described using the movement notation. The two girls do not explicitly follow each other but seem to affect each other in movement. Their kinaesthetic engagement enters the realm of the mathematical understanding of graphs, and in exploring the meaning of the lines, even the smallest perception about moving oneself might matter. I do not consider Vittoria's claim as a wrong statement that was suggested in the discussion, but rather as one of the ways in which movement manifests itself in the interpretation of graphical representation. Being more than one and affecting each other somehow become depending on one another. Riccardo's outsider eye recovers the experience and the different positions of the two girls at the same time ("Then, when Sofia was, when Sofia was close, she [Vittoria] did not necessarily move close too"). Riccardo compares the two positions and movements by bringing forth the possibility of independence between the lines in virtue of the presupposed independence of the two girls. On the one side, movement is that which connects the two experiences; on the other side, movement creates the possibility for the independence of the lines as well as of the people to be part of this assemblage of lines and girls.

Then, the focus of the classroom discussion shifts to two questions: one question specifically refers to a tiny portion of the graph created by Vittoria; the other instead refers to the movement of the limbs and its role and involvement in the creation of graphical representations.

Thus, Sofia first directs attention to a specific part of Vittoria's graph (that highlighted in Figure 6.12), adopting a local point of view (Maschietto, 2008). It is only a little portion within the entire composition of the graphical lines. Nevertheless, Sofia expresses the desire of concentrating effort in understanding how that "little wave" originated on the screen. Her whole body is sustaining the suddenness of the changes that occur to the line and that sustain the tiny shape, as she bends the knees ("then right below") and quickly rises with intensity ("then right above"). The little piece of graph is reimagined inside a short duration, therefore Sofia's movement carries with it the temporality ascribed to the graph portion that is under investigation. The qualitative nature of the line catalyses the activity and emerges from Sofia's bodily engagement at the IWB.

Secondly, Gianluca and Michelle investigate the role of the arm's movement in using WiiGraph. Gianluca's perceptual attunement to the experience of moving is captured by his bodily engagement while he refers to an apparent jagged portion of the line (see the gestures in Figure 6.14d). In fact, he changes his posture to a murky one and bends over, then slightly and repeatedly moves his wrist from left to right as if he was holding a remote with a shaking hand (Figure 6.14c; superimposed images with changed transparency filter). Then, he rises in a resolute but smooth manner when he contrasts the previous zigzag with a smooth line and, in turn, a non-movement of the arm. The jagged line on the screen and the potential smooth line actualised by Gianluca's gestures are articulated along the lines of the possible influence of the arm's movement on the qualitative aspects of the line. Through imagining the line conditioned by the little movements of the wrist that holds the remote, the graph acquires a qualitative nature in response to a more specific movement: that of murky and dark, even disturbing appearance. Instead, the smooth line is provoking a smile and the resolute gesture in front of the class.

After Gianluca, Michelle also stresses that what matters in using WiiGraph is not just the whole-body motion, the back and forth macro-movement of the two girls, but the local, micro-perceptual limbs' movement. Other qualitative aspects of the lines emerge as potentially relevant in that the body is a centre of indeterminacy (borrowing from Bergson, 1896/1988) and can be seen as an expressive field of virtuality for the mathematical graphs.

In this episode, through the discussion of Sofia and Vittoria's exploratory experiment, I have investigated how exploring becomes playing with qualitative alterations of the given situation and imagining, in the already given scenarios, potentialities that perturb the past experience.

6.3.3 Upper secondary school: Grade 10

In the grade 10 intervention, the first volunteers were Alina and Silvia (Figure 6.15). They came to the interaction space and were given the controllers and the same general instructions about how to point to the sensor and the coloured dots on the screen. Differently from the other classroom interventions, the students were also told that they could initiate a session by pressing the button A of the remote, which is located in the upper part close to the thumb position (when grabbing the controller).

Despite this instruction, in the first trials the girls kept the button pressed while moving in space, so nothing happened, even if they moved back and forth (no lines were created on the screen). After overcoming these technical problems, the graphs finally began originating on the screen and the exploratory experiment occurred (Figure 6.15).

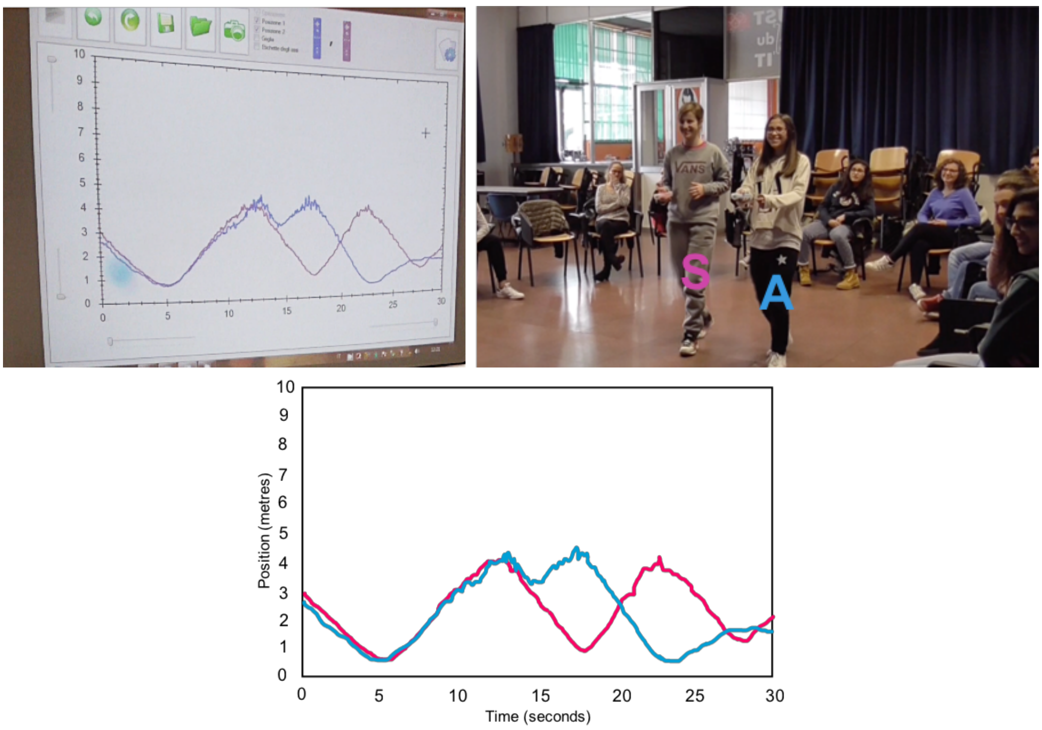


Figure 6.15. Alina and Silvia's exploratory experiment and their graphs

I propose a notation for the experiment of Alina and Silvia in Figure 6.16.

Figure 6.16. Movement notation for Alina and Silvia's experiment

As in the previous subsections, the notation aims at grasping the experience of movement by focussing on the micro-movements that populate the experiment and sustain the emergence of the particular graphical configuration. In this experiment, the two girls begin by moving side by side, nearly at the same pace, even if with their own walking rhythm. After about 15 seconds from the start, Alina suddenly differentiates her movement by stepping backwards while Silvia is still stepping forward. As they move towards each other after few seconds, right before crossing each other while moving in opposite directions, Alina jokingly utters “Uuuuh”, and the two girls gaze at each other at slightly different times. In the second part of the experiment, they seem to move much more independently from each other, even though they *look for each other* (as we can notice from both the notation and the graph).

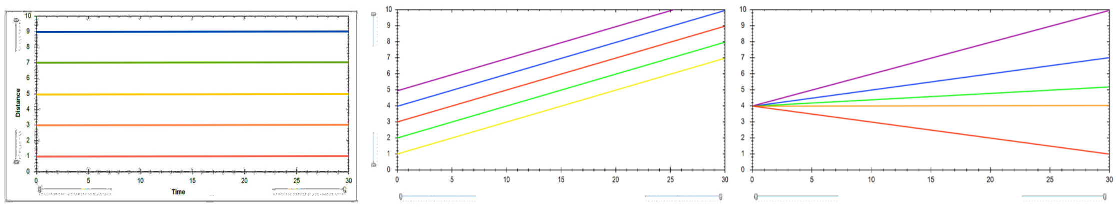
They do so literally, in the sense that each of them gazes at the other. Additionally, they look for each other in the sense that they seem to implicitly coordinate their movements as the changes in direction happen almost at the same time (around 17 and 23 seconds). The two girls giggle during the experiment at different times and it is hard to say with certainty whether this is much more related to jitters or to their enjoying of the situation. Both these affective forces probably sustain the movement of the students, from the initial parallel movement to their insistently looking for each other while they move in different directions. The experiment therefore sheds light once more on the peculiar characteristic of moving with WiiGraph, that is, being two in the interaction space. This characteristic fosters the recognition of the other, even in the form of an abrupt differentiation or in the form of a joint movement, but always implicates some kind of relationship with the other. In Alina and Silvia’s experiment it is apparent in the ways in which the two girls modulate their intensities and are moved to move in space, as I have expressed through the notation and described in this brief discussion.

The appreciation of the feature of being two through the small details that connote the movements inside a motion experiment brings forth the need for integrating into the experience of using the tool a wider spectrum of perceptual and affective tones that populate the interaction. It is no coincidence that, in the collective discussion that follows the experiment, the participation of Alina and Silvia is permeated with recollections of the qualitative alterations of their movements. Silvia stresses that “at some point, maybe, she [Alina] accelerated, and I decelerated” and Alina adds “and maybe I changed direction”. The graphical representations are in this way infused with the relationships that are perceptually experienced in movement, which come to characterise the graphs themselves as mathematical relationships.

Developing a tool perspective (Nemirovsky et al., 1998) involves addressing the use of a tool with particular care to some aspects, while others might be partially shaded. In the case of mathematical instruments like WiiGraph, the adoption of a tool perspective brings the emergence of technical, bodily and mathematical aspects, which are all at play in using the tool. In the episodes of this section, I have investigated how the students in our study encountered WiiGraph by means of an exploratory experiment. In the analysis, I brought forth a detailed and evocative description of the encounters of the students with the technology, which treats it as part of thinking in movement. Using a tailored-made movement notation I have highlighted how freedom of movement and affective attunement to the qualitative nature of movement constitute the exploration as a unique event. The use of a notation enlarges the field of what is significant and help us understand the movement as a notation for the graph, in that its qualitative nature forms the experience of moving and is not completely captured by the graphical representation. By describing how WiiGraph was envisioned in following moments of collective discussion and described in the written tasks, I have also pointed out how caring about the queer aspects as well as mostly unnoticed parts of the graphs become significant when we turn to movement and shed some lights on the process of moving-thinking in the use of WiiGraph.

6.4 Ways of moving, ways of thinking

In this section, focus is on three graphical representations (Figure 6.17) that have been chosen for a parallel activity across the various grades (a methodological choice in the design of the teaching experiments). All the representations show a Cartesian plane, similar to the one given in the window of WiiGraph, on which five lines are drawn, which belong to a particular family of straight lines: five horizontal straight lines (Figure 6.17a), five parallel slanted straight lines (Figure 6.17b) and five concurrent lines (Figure 6.17c). The three configurations, which cannot be created with WiiGraph that only gives two lines at a time (when using *Line* without any target), were introduced with the second worksheet (Scheda 2) respectively to the primary, lower secondary and upper secondary students who participated in the study.



(a) (b) (c)

Figure 6.17. The three families of straight lines

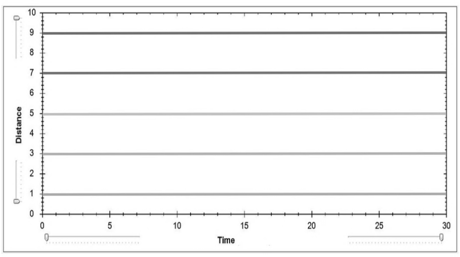
In this chapter, particularly in this section, attention shifts to the written activity developed for each of the configurations, the bodily strategies that the students brought forth in relation to them, the assembling of meaning that was implicated, the concept of function which emerged out of the specific activity and the ways in which the different configurations merged into each other during the collective classroom discussions.

6.4.1 Horizontal straight lines

Primary school students investigated how to create parallel horizontal straight lines with the remotes first through an unexpected experiment in the first day, then during the second day, when they positioned each remote on a chair to produce two very "precise" horizontal straight lines. They also compared the relative positions of the chairs and the remotes with the relative positions of the lines on the IWB (below or above according to their distance from the sensor). After the collective discussion, the class was divided into groups of four students and faced the second worksheets (Scheda 2). The written task of the worksheet is the following:

> Imagine that you have 5 controllers at your disposal and that you see these graphs on the IWB: (Figure 6.18a).
>
> Explain the way you would create them.

The groups of students produced different diagrams that I discuss in the following, which relate to the five horizontal straight lines given by the task. In Figure 6.18b we see a group of children, who share their ideas (in the collective discussion after group work) and negotiate their actual position in the interaction space.

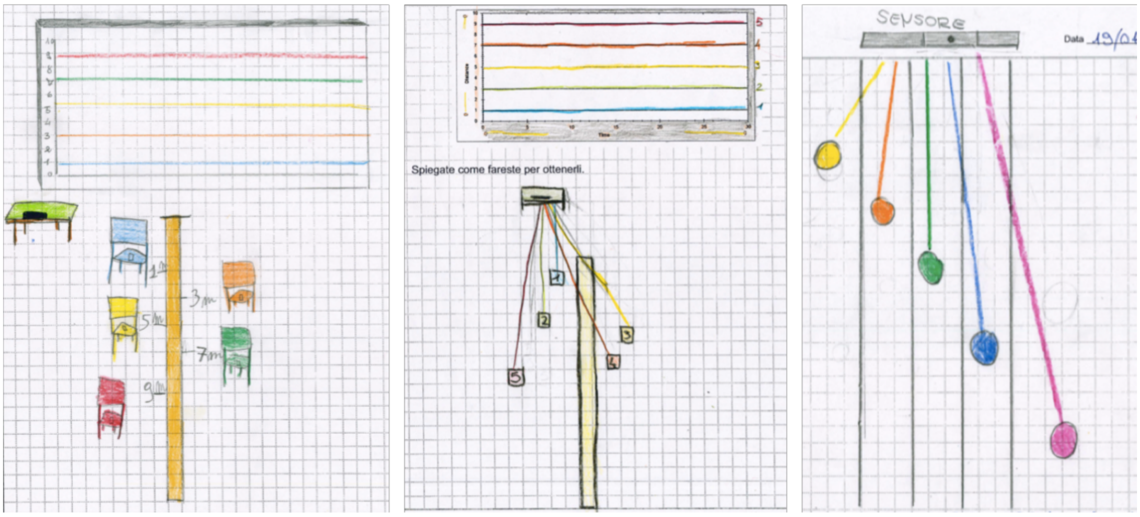


(a) (b)

Figure 6.18. (a) The graphs given at primary school (black-and-white format); (b) children positioned in space as if they were creating those lines with WiiGraph

Being more than two

The task asked the children to explain which strategies in terms of movement they could elaborate on a situation that is similar to those experienced in the classroom, but that entails a higher number of lines and a specific relationship between those lines (in this case, 2-meter distance from each other). From the written productions of the groups we can observe the use of different diagrams and notation for capturing this relationality. All the groups realised that the remotes have to be placed in different positions, that is, at different distances from the sensor, but they captured this aspect in different ways. Three out of the six groups in which the class was divided drew a diagram that connects the positions of the controllers with the corresponding lines, using colours (see Figure 6.19).



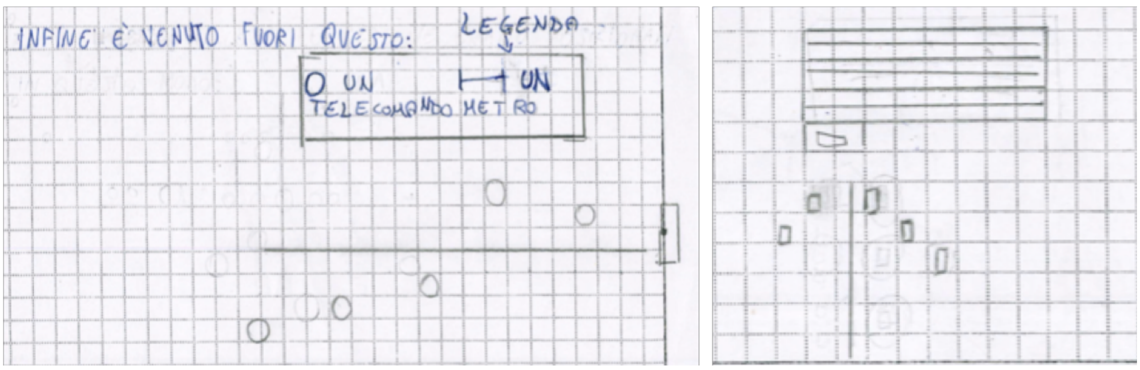
(a) (b) (c)

Figure 6.19. (a) Group 1, (b) group 2 and (c) group 3's diagrams in Scheda 1

Note that the given configuration was black-and-white, exactly to give the students freedom to act on it, and colour is only one of the eventual variables that could be adopted for a strategy in this context, for example to distinguish the various lines, even before matching them with suitable counterparts in the interaction space. Groups 1 and 6 suggested the use of chairs in their written productions; group 1's drawing also identified the correspondence between each line and the chair by means of the same colour used to depict the chair (Figure 6.19a). Diagrams of groups 2 and 3 used dots or little squares to capture the positions and to connect them to the sensor with a line (Figures 6.19b and 6.19c). Groups 1 and 2 also added numbers (1, 3, 5, 7, 9) next to the chairs or inside the squares that mark the positions in the interaction space, in order to identify the height of each line (Figures 6.19a and 6.19b).

All these diagrams, as well as those of groups 4 and 6 (Figure 6.20), take into account an important technical issue related to the use of WiiGraph, namely the fact that the remotes' signals have not to be interrupted by the presence of another body. In the case of only two children or remotes in space, it was sufficient to place each user on one side of the tape strip on the floor, in order to avoid the alignment that influences interference. Instead, here the situation is quite different. The diagrams show how the children negotiate the positions in space, placing three of them on one side and two on the other side (even in different orders: Figures 6.19a and 6.19b, Figures 6.20a and 6.20b), or creating 5 corridors, one for each user (like in Figure 6.19c).

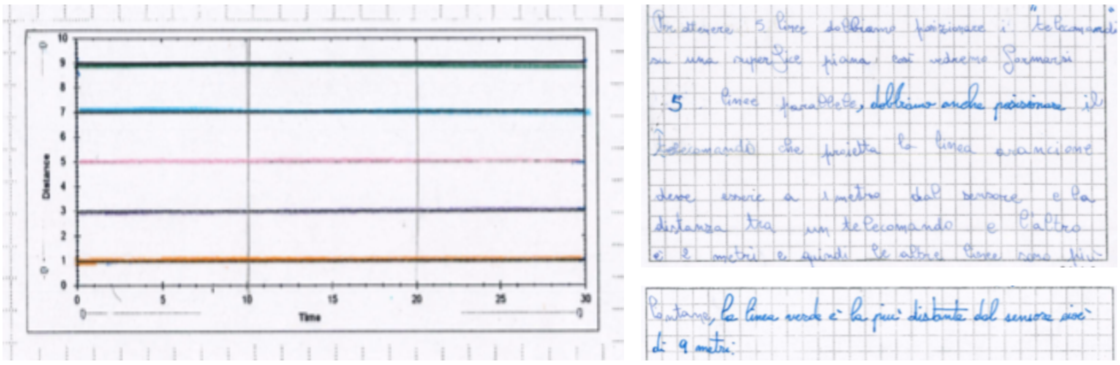
The diagram of group 4 also uses a legend (O a remote, |—| one metre) to capture the relative positions of the remotes (it is not clear, though, how the students used the unit for distance in the end; Figure 6.20a). In Figure 6.20b, we see that, even if in the written argument the students of group 6 recognize that "it is necessary to use 5 chairs (2 metres apart), with a remote on each chair, obviously steady", in the drawing two pairs of chairs are horizontally aligned (i.e., at the same distance from the sensor approximately).



(a) (b)

Figure 6.20. (a) Group 4 and (b) group 6's diagrams in Scheda 1

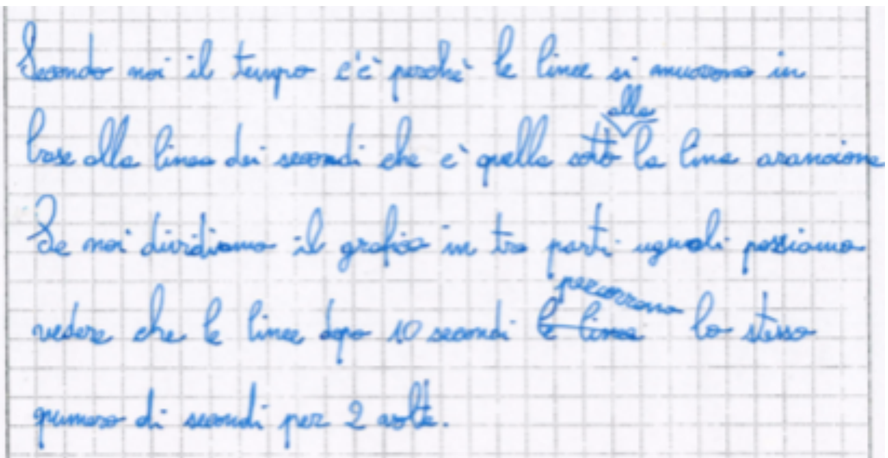
The students of group 5 coloured the lines (see Figure 6.21a) but further differentiated themselves from the other groups by not using any drawing to represent the users' position in space. They instead explained that "the remote that projects the orange line has to be at 1 metre from the sensor and the distance between one remote and the other is 2 metres and therefore the other lines are further, the green line is the furthest one, that is, 9 metres far" (Figure 6.21b).



(a) (b)

Figure 6.21. (a) Time intervals and (b) fragment of explanation in group 5's worksheet

These students interestingly drew two vertical lines in the given Cartesian plane corresponding to the numbers "10" and "20" on the horizontal axis (Figure 6.21a again), therefore dividing the plane into three equal parts, and developed a discourse on the relevance of time in this representation (see Figure 6.22).



For us, there is time because the lines move according to the line of seconds, which is the one under the orange line. If we divide the graph in three equal parts, we can see that the lines, after 10 seconds, cover the same path of seconds 2 times.

Figure 6.22. Final part of group 5's explanation

Discussion

The students worked in groups to analyse the given situation, which purposefully offered them a bundle of lines that cannot be obtained with the software, but that adhere to the same conventions. For example, most of the groups outlined through their diagrams the relevance of the remotes' spatial disposition to avoid light interferences (like the displacement of chairs, squares or dots in the imagined interaction space), which might affect the appearance of the lines in using WiiGraph (the children in fact have experience these gaps and flaws). This aspect enters into the strict relationships between the remotes' positions (two metres apart, as all the students realise) and the constraint of non-movement to create horizontal straight lines. In order to deal with this constraint, some children proposed to use stable surfaces, like that of the chair, which they already experienced as a possible solution to obtain "precise" (smooth) horizontal lines.

The last group we presented, group 5, struggled with the understanding of the role of time in the graphical representations and argued that the horizontal axis is effectively "the line of seconds". By dividing the given Cartesian plane in three parts the children thought of the lines as covering the same space three times. It is interesting to notice here that, even though the representation does require that the movements that model the situation are, in fact, non-movements, the students put the lines into motion to recover their dynamic dimension. The movement is "of the lines" in the sense that they originate at the same time, and the relative movements are discussed more in terms of *lines that move* rather than in terms of relationships between movers or controllers' movements. Even if the English word "Time" appears under the horizontal axis in the graph area inside the worksheet, in the collective discussion the students did not yet agree about the role played by time in modelling the situation. At this point of the teaching experiment, for most of them it is not true that time matters (as few, though, argued already).

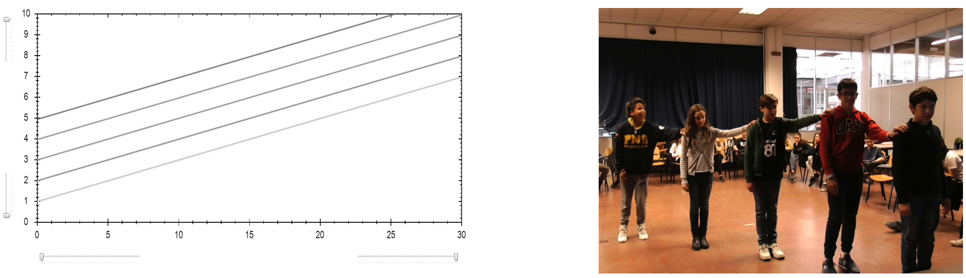
To say it differently, for the children it is not that a line stops moving at the end of a motion experience because (modelling) time ends but rather because the available space "is finished". According to the written text of group 5, instead, in the graphical representation time *exists* thanks to the fact that *space unfolds by means of lines' movements*. This is a precise ontological shift in the work of the students towards understanding graphs, a move that exploits movement as the means by which interpreting the unfolding of time. It is also an aspect that consistently informs the pivotal importance of making sense of horizontal straight lines in graphing motion activity, already discussed in the literature (e.g., Ferrara & Robutti, 2002). What is at play here is the recognition of a different kind of movement, a movement of the lines, which engenders the sense of duration and creates a global vision of the spatio-temporal relationships. The recognition is detached from a point-wise interpretation of the Cartesian plane, and rather unfolds through the cutting of lines along three equal intervals, which "cover the same path of seconds" (Figure 6.22).

6.4.2 Parallel straight lines

During the second day of the grade 7 intervention, the class worked in groups on a written task similar to that presented in the previous subsection. The worksheet (Scheda 2) asked the students to work with parallel non-horizontal straight lines (like the ones in Figure 6.23a, again in black-and-white format), a particular family of straight lines.

The given task was the following:

> Imagine that you can use more than two controllers and you create five straight lines: (Figure 6.23a).
>
> 1. Explain which movements can generate these lines, paying attention to specify all the information that is relevant for you.
>
> 2. In your opinion, what do the five lines share? Instead, how do they differ? What about the movements? Explain your reasoning.



(a) (b)

Figure 6.23. (a) Given graphs (black-and-white format); (b) students' coordination to create parallel slanted straight lines

The written activity proposes a Cartesian plane that show five parallel slanted straight lines to be investigated from the point of view of both the corresponding movements that could generate them and their relationships, pushing attention towards similarities and differences (an example is shown in Figure 6.23b). We expected the students to observe that the lines are parallel to each other and to reason about this aspect in terms of absolute and relative motion (constant speed, equal for each of the movers, but different starting positions).

The second task aimed at complementing the first one by paying attention to those aspects that are preserved for the lines/movements and to those that are not. I first analyse the main issues that emerged from the students' answers and then how these issues were tackled in the collective discussion, which followed the group work.

The given configuration might seem not particularly challenging as it presents "the same line" translated vertically four times. Nevertheless, it is quite interesting to notice the ways in which the students dealt with it, especially regarding the movers' speed.

In the following, I take the answers coming from some of the groups, which I will then analyse according to three main lines of thought: (1) qualitative aspects of motion; (2) speed of motion; (3) strategies of movement. Figures from 6.24 to 6.28 show excerpts of the worksheets of groups 1, 6, and 7.

Group 1

LE 5 PERSONE DEVONO PARTIRE DA 1 METRO DI DISTANZA FRA CIASCUNA
IL 1° È A UN METRO DAL SENSORE.
BISOGNA INDIETREGGIARE MOLTO LENTAMENTE RIMÉNDO ALLA COSTANTE DISTANZA DI 1 METRO

The 5 people have to start from 1 metre-distance between each other. The first is at one metre from the sensor. It's necessary to go backwards very slowly, keeping a constant 1 metre-distance.

LE CINQUE LINEE SONO PARALLELE.
HANNO DI DIVERSO LA DISTANZA DAL SENSORE.
DEVONO MUOVERSI TUTTI E 5 ALLA STESSA VELOCITÀ, MOLTO LENTI, IN MODO CHE IN 30 SECONDI OGNUNO PERCORRA 6 METRI
DEVONO FARE GLI STESSI MOVIMENTI PERCHÉ ALTRIMENTI NON SONO PARALLELI.

The five lines are parallel. They differ over the distance from the sensor. They, all 5, have to move at the same speed, very slow, so that in 30 seconds everyone travels 6 metres. They have to do the same movements because they're not parallel otherwise.

Figure 6.24. The written answers of group 1

Group 6

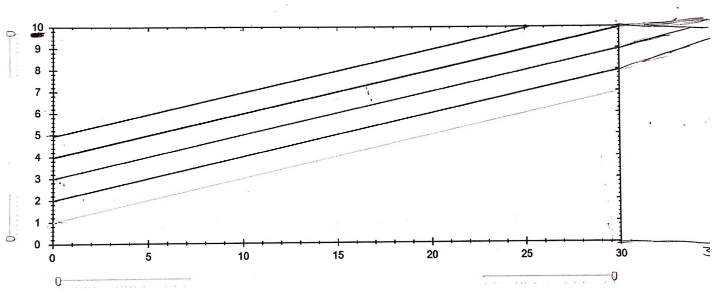
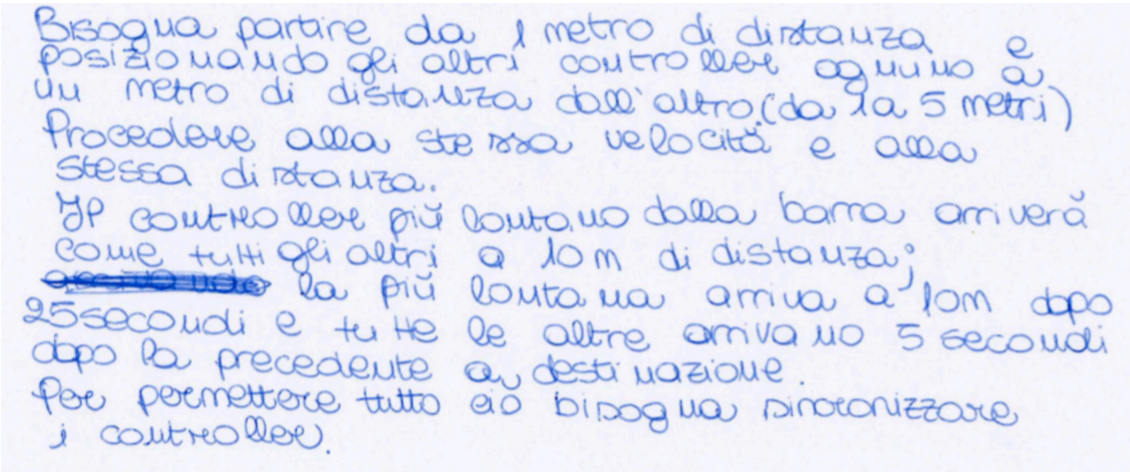


Figure 6.25. Diagram from group 6: the lines are extended to the 10 metre-height



It is necessary to start from 1 metre-distance and positioning the other controllers one at a distance of one metre from another (from 1 to 5 metres).
To proceed at the same speed and at the same distance.
The controller that is the furthest from the bar will arrive like the others at 10 metre-distance; the farthest [line] arrives at 10 m after 25 seconds and all the others arrive at destination 5 seconds after the previous one. To allow all of this it is necessary to synchronise the controllers.

Figure 6.26. The written answer of group 6

Group 7

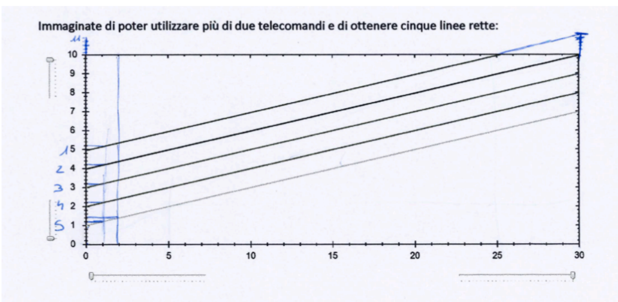
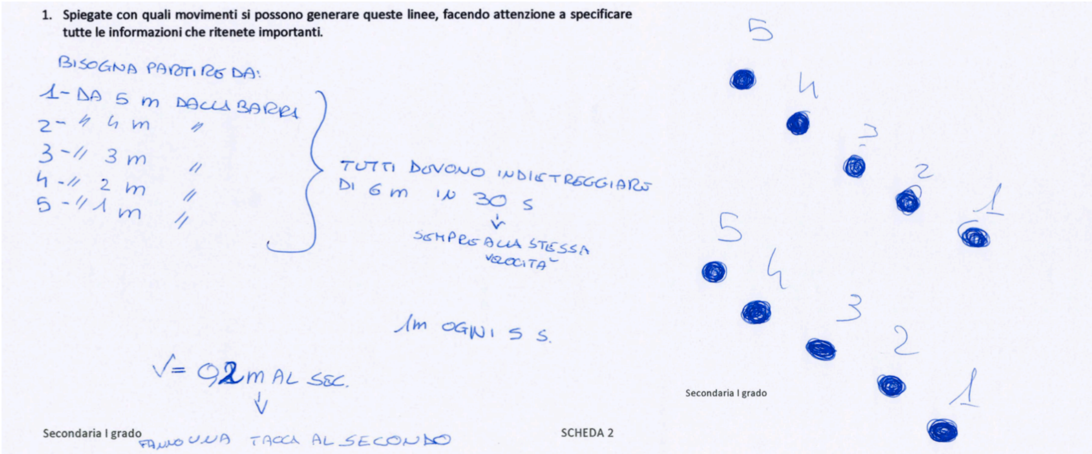


Figure 6.27. Diagram from group 7: the upper line is extended to 30 seconds



Everyone has to go back 6 m in 30 s
→ always at the same speed
1 m every 5 s
v = 0,2 m/s → they do one mark every second

Figure 6.28. From group 7's worksheets

Discussion

In their written answers (many of which I am not going to analyse in this section), most of the groups furnished descriptions of the movements that can originate the five parallel slanted straight lines by means of their starting positions and speed. The starting positions have to be different, while speed has to be the same for everybody moving and to remain constant for the entire duration of the experiment. I chose to present the written productions of groups 1, 6 and 7 to highlight how ideas complementary to this descriptive information about the graphs emerge along (1) qualitative aspects of motion; (2) speed of motion; (3) strategies of movement. For example, group 1 described the movements as "to go backwards very slowly" (Figure 6.24) and this is relevant for all the actors, who have to move at the same speed, covering 6 metres in 30 seconds. Introducing this numerical relationship for the required speed, the group again refers in the written to the use of the qualitative adverbial expression "very slow". In terms of the strategy implicated in this specific configuration, the students wrote that everyone has to perform the same movements, otherwise the lines are not parallel, and thus they invoked a kind of coordination among the movers to obtain the desired relationship. In addition, at the beginning, the students stressed the importance of "keeping a constant distance of 1 metre" from one another, but this aspect is not explicitly interconnected to the parallelism of the five lines.

Groups 6 and 7 in some sense tackled the issue of same speed from opposite but complementary perspectives.

While group 6 imagined that the experiment lasts until all the lines arrive at 10 metres (Figure 6.26), group 7 extended one of the lines so that they could spot at which height the line arrives after 30 seconds. The first diagram (group 6, Figure 6.25) suggests a vision of speed that implicitly considers a specific route to be travelled as necessary for a comparison between movements. The difference in the initial position and the constant distance, which has to be preserved over time, are changed into the delay in the arrival position (ten metres far from the sensor). Speed does not emerge by comparing the travelled distance over the same amount of time (as, for example, it occurred with group 1): the movements' speed is the same as the delay in the arrival position is constant and linearly depends on the relationship with the starting positions. Instead, group 7 showed that the highest the line, the highest the arrival point. This "last line" is prolonged above the space delimited by the Cartesian plane and reaches a distance of 11 m after 30 seconds (Figure 6.27). It is as if the students imagined that the relationship between movements is still maintained, even if the software is no longer able to capture it inside its window. The new addition on the given diagram in this second case, and in the first case as well, captures the force of the diagrammatic activity in breaking with the conventional schemas that are attributed to the Cartesian plane and expands the field of the experiential ground towards the first definitions of speed in spatio-temporal graphs.

Group 7 also investigated the situation by means of another diagram, which seems to stress the relative positions of the people: these relative positions are preserved at the beginning as well as at the end of the motion experiment (Figure 6.28). Another group, group 3, during the written activity used five coins to represent the movers and a pen to capture the bar (Figure 6.29). The group has to deal with the ambiguity between the expression "the same distance" used by one student, Riccardo (see again Figure 6.29). Riccardo remarks that saying that "the five people have to be at the same distance" do not refer to the distance of each person from the sensor, but to the distance in-between two people, which necessarily has to be preserved in movement. This similarity in tackling the task is relevant because both the diagram with translated dots and the use of moving coins in the group activity deal with the intrinsic ambiguity of the situations, in which a bundle of distances is involved. Mainly, in the students' discourses and written productions we can distinguish two different 'types' of distances: that of each people (or remote) from the sensor and the relative distance from one person to the next. In the Cartesian plane, by adhering to the hierarchy dictated by the reference system, we can assume the primacy of the former distance. In the activity, though, the latter gains a leading role for understanding how the relational movement of the five people is sustained.

Figure 6.29. Group 3 uses coins to display relative positions and movements

From the collective discussion

During the collective discussion that took place after group work, five students are asked to move as if they were producing the five parallel slanted lines of the task and related aspects are discussed with the researcher.

The students that volunteered for this new task begin discussing about their bodily strategies to enact for imaginarily obtaining the translated graphs. They should move as if they were using WiiGraph, although they are not using it. Initially, they agree about starting at 1 metre-distance from each other and moving one tile at a time. Moreover, they decide that their positions in front of the sensor should be slightly horizontally shifted as if they all had to point their remote to the sensor bar (an issue already present in the primary school diagrams we analysed in the previous subsection). No one of them is holding a remote, but they all adopt a tool perspective. A first experiment starts: the five students step some distance backwards by carefully looking at the floor to check their distances. However, they move at different times and many classmates claim: “It’s not right!”.

Figure 6.30. Second attempt: the students coordinate with each other by looking at the tiles on the floor while the classmates count to ten

The researcher therefore suggests that the classmates count to ten while the five people move in space, as if the experiment only lasts 10 seconds in total. A new experiment starts (Figure 6.30): the five students step the interaction space (backwards) following the dictated rhythm and looking down on the floor. When the class gets to count “six”, the student who was at the end of the line (Michelle) reaches the end of the available space and is forced to stop moving. In few seconds, all the other four students are aligned in a row. Most of the classmates rise up with disapproval. Sofia proposes that “When Michelle [the farthest one from the sensor] gets to the end, all of them have to stop”. The students discuss the possible configurations that might result from the movements they just performed, and how they differ from the desired one. Alessio suggests that they should do “less than a step per second”. The five students continue to discuss on how to step space backwards (“half a tile”, “less than a tile”). The researcher thus triggers a new approach by asking “Isn’t there a way of having one person leading without deciding *a priori* that you will do half a tile per second?”.

After further discussion, a last experiment starts (Figure 6.31): the first student in the line leads, while the others follow him. They try to keep relative distance fixed from each other by maintaining the right hand on the shoulder of the mate who is ahead, with the arm straight and stretched. The classmates count to ten again, and the five students still look at the tiles to modulate their steps.

Discussion

In this last part of the episode, we see the students struggling with the ways of moving that could create five parallel slanted lines. These ways encompass:

- The rhythmic pattern dictated by the classmates through sounds, clapping hands and repetitive beats in counting to ten;
- the arm placed on the shoulder of the mate ahead;
- one student (the first one in the row) leading the troop.

These ways of moving and coordinating movements are significant in making sense of the written activity as they make room to consider the way that the transformation applied to the functional relationships holds. When the transformation turns to the body, what matters is not only the general way of doing something but the consideration of all the complex nuances that add meaningfulness to the choreographic and coordinated movement.

Figure 6.31. Coordination achieved using arms, sounds and tile reference

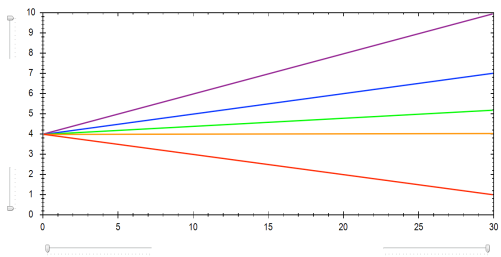
I have presented an activity which was deliberately mathematical and that involved multiparty whole-body interactions in the classroom setting, similarly to those discussed in Ma (2017). In our activity, the students' bodies are moving in a coordinated effort to find ways of preserving some specific mathematical relationships, that is, the vertical translation between one and the other line. This is achieved by means of negotiations by a complex arrangement of bodies. The significance of the activity relies precisely on the new bodily strategies that are discovered and explored after the written worksheets, and on the ways in which these are shared and lived with the collective body of the class.

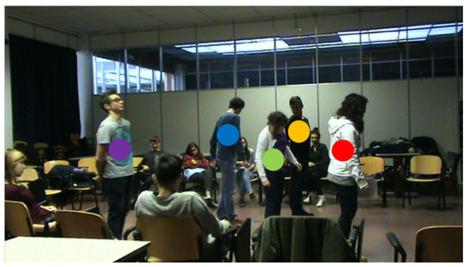
6.4.3 Straight lines

In this subsection, I discuss the activity of grade 10 students around a written task about five concurrent lines (Figure 6.32a), a new family of straight lines, which we can conceive as a variation of the previous bundle of parallel straight lines simply imagining of stretching those in different directions while making them collapse in a single origin.

The task (Scheda 2) was the following:

> Imagine that you can use more than two controllers and you create the following graphs: (Figure 6.32a).
>
> 1. Explain which movements can generate these graphs, paying attention to specify all the information that is relevant for you.
>
> 2. In your opinion, what do the five straight lines share? Instead, how do they differ? What about the movements? Explain your reasoning.





(a) (b)

Figure 6.32. (a) Cartesian plane given in Scheda 2; (b) five students who coordinate with each other to produce movements for the five concurrent lines (coloured dots correspond to the lines)

From the written productions, we get a sense that the grade 10 students dealt with ease with the issue of speed for each of the different movements: they interpret it as constant, because the graphs are all straight lines; they calculate the speed through the ratio between travelled space and experiment duration[1] (e.g., see Figure 6.33).

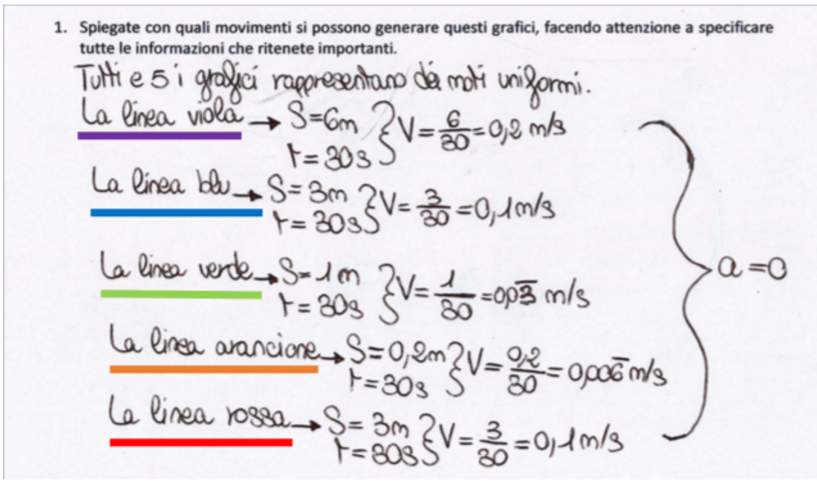
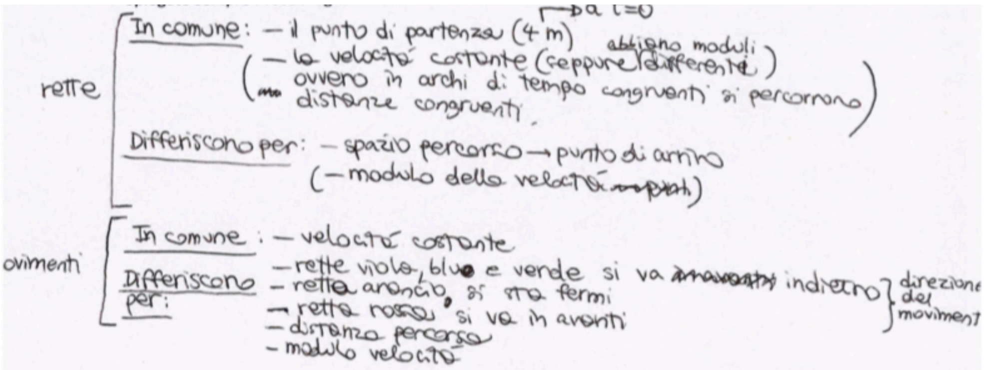


Figure 6.33. Like most of the groups, group 5 calculates the speed 'for' each line. Before each arrows the students write "the line" plus the corresponding colour they refer to (coloured underlines have been added to trace this correspondence)

All the groups also recognize that the red and blue lines share the same speed/travelled space even if the movements that could generate these lines have to occur along different directions, while all the other movements differ in speed and from them. Especially in relation to the second point of the task, most of the groups of students do not clearly separate the characteristics that are "of the lines" and those that are "of the movements". For example, group 4 (Figure 6.34) articulates the written answer to the second question by distinguishing these aspects (lines and movements; in common and different elements) spatially, while in fact they are all tangled up in the writing.

This overlapping might be interpreted in terms of fusion (Nemirovsky & Monk, 2000), as the features of the lines and those of the movements clearly do not have neat borders but fade into each other. In speaking of the speed of the lines, the students are not explicitly referring to how the lines 'move' in the Cartesian plane, but rather are already considering the slope of the lines in terms of speed in the modelling situation.

We see this very clearly in the work of group 4, in which the different points are articulated in a way that pushes towards a definite separation, but they still emerge with many conjunction points. For example, the expression "constant speed" appears in reference to both the lines and the movements. In explaining what the constant speed means in reference to the lines, the students write: "that is, in congruent time intervals congruent distances are travelled" (Figure 6.34 again). Therefore, at the same time, the travelled distance, which speaks directly to the people's movements, is instead used in reference to the lines. Since the phenomenon of fusion has been already investigated in the literature, it is not the focus of our discussion. I just want to point out that, as Nemirovsky and Monk (2000) observe, fusion even characterises the practice of more expert mathematics practitioners. In this case, the grade 10 students already have some experience, especially they consistently use symbolic expressions and know how to calculate explicitly the speed of the movement, given a corresponding distance versus time graph. Following these researchers, what is significant is what experiencing fusion brings to the situation. Therefore, to investigate this aspect further in our context, and to give an original contribution to this line of research, I will complement the discussion of this episode stressing the importance of moving oneself as the very meaningful and interesting counterpart of verbally and algebraically interpreting movement.


rette
In comune: — il punto di partenza (4 m) → a t=0
— la velocità costante (seppure abbiano moduli differenti)
(ovvero in archi di tempo congruenti si percorrono distanze congruenti.)
Differiscono per: — spazio percorso → punto di arrivo
(— modulo della velocità)
movimenti
In comune: — velocità costante
Differiscono per: — rette viola, blu e verde si va indietro } direzione del moviment
— retta arancio, si sta fermi
— retta rossa si va in avanti
— distanza percorsa
— modulo velocità

Lines. <u>In common</u>:
- the starting point (4m) → at t=0
- the constant speed (even if with different *moduli*), that is in congruent time intervals congruent distances are travelled

<u>They differ in</u>:
- travelled space → arrival point
(- speed *modulus*)

Movements. <u>In common</u>:
- constant speed

<u>They differ in</u>:
- purple, blue and green straight lines, one goes backwards {movement direction
- orange line, one stands still
- red line, one goes forward
- travelled distance
- speed *modulus*

Figure 6.34. Group 4's written answer to the second task of Scheda 2

Like it happened in the case of the five parallel straight lines (§6.4.2), the students are asked to coordinate with each other as if they were creating those lines with the remotes. Five students volunteer.

At the beginning of the discussion, one student, Arianna, takes the lead and summarises:

> Arianna: We all have to start from the same position of four metres from the distance at which the sensor starts capturing the signal. There will be a person, who will stand still, who will be a kind of reference point for the others, because for anyone who gets closer to the sensor, the line will have a negative tendency, that is, will tend to go down. Anyone who goes far from the sensor will have a tendency that will tend to increase, because she is getting further from the sensor. We have to move at a constant speed, [...] constant speeds, but each one different from the other.

For the following eight minutes, the students struggle with adjusting their speeds in accordance with the new constraint given by available space and the difficulty of reproducing extremely slow speeds with their bodily movements. In the video data, this segment of classroom discussion is quite messy and chaotic for the ways in which the students interact and argue about the movements. Presenting a transcript of it would take our discourse far from the original intent of the section. Nevertheless, it is worth focussing on this event to highlight that, even though the students investigate the situation analytically with relative ease in the written worksheets, as they turn to perform choreographies of movement, they experience sincere struggle and are forced to change their perspectives. For example, they take tiles as a reference, so that what was one metre in the given situation now refers to one tile on the floor. In addition, each of them has a specific number of tiles to be covered in the given time. For some the displacement is ridiculously small, therefore the researcher suggests reducing the experiment to ten seconds. The discussion further expands with someone who observes that speed will change if they do not calculate the displacement proportionally with respect to the reducing of time. A first experiment with five actors occurs. They all start from the same position and each is responsible for a specific speed, while the other classmates count to ten. At the end, Arianna admits that she covered less space than expected. Two other experiments are conducted. The student who has to stand still is taken as a reference point, while the others move along a spectrum of variation that depends on their speed.

The activity closes with a final discussion on the role of slope in the recognition of different speeds in the graphs. This aspect is also pivotal to understand speed as the parameter that relates and, at the same time, distinguishes the lines of the given family.

I am not arguing that moving is *necessary* to understand or to solve the task, but that the qualitative spectrum that is opened by/with/through such investigations of mobile relationships enlarges the field of possibilities in the given activity, making the students confront with that multiplicity of interactions and potentiality that involve their bodies.

In seeking for coordination or for the most suitable way of moving at a certain speed, it is not that the students learn something different. Rather they experience a hidden counterpart of what they already know. Again, this is not necessary for mathematical competence to be acquired, but creates new nuances in the ways in which, for example, the slope of a line is perceived and grasped as the parameter that potentially transforms the movements into each other. It is really speed that differentiates the person who stands still from the others.

Drawing attention to the experience of moving oneself in coordinated efforts with others, I do not want to claim that it is a skill that needs to be mastered. Instead, I simply want to stress that the experience of moving oneself involves traversing a mathematical landscape as if one was travelling inside a material, feeling obstructions and resistances, as well as encountering the fluidity of relations that is intuitively given to us to when we are attuned to the qualitative dimension of movement.

6.5 When lines cross, when people meet

In §6.2.2, we moved the first steps in investigating how the event of crossing lines in WiiGraph might be pivotal to make sense of the graphical representations. This section intends to enlarge the view around this aspect, which was central in the pilot study, in the design of the interventions concerning classroom activities and tasks, and in the students' experience, as the episodes below will bring forth.

6.5.1 They cross each other!

During the first day at primary school, after the exploratory experiment we have discussed in §6.3.1, the students formulated hypotheses on what the lines represent. The first major hypothesis, which was shared among the children, concerns the behaviour of the line, which moves up or down whether one moves backwards or forward (facing the sensor bar). One student, Luca, explicitly said that this is due to the fact that the line represents a distance; the rest of the class reluctantly agreed but seemed unconvinced. Therefore, the researcher asked the children to think of an experiment that could help them establish whether Luca's conjecture is correct. After almost ten minutes of discussion led by the researcher, the children planned to set up an experiment in which two children (Giorgia and Dario, who were quickly told to go to the interaction space; Figure 6.35a) started from different positions (one closer to the sensor with respect to the other) and then they moved 'exchanging' their position, that is, the one who was closer to the sensor moved backwards, while the other moved forward. We go back to the moment when the children began discussing about the kind of experiment to conduct.

Giorgia and Dario are in the centre. Dafne is the first to propose: "While Giorgia goes backwards, at the end of the bar [the tape strip], Dario goes forward" (Figure 6.35b). "And, then?" asks the researcher. "They cross each other!" says Federico, who relaunches on the two movements of Giorgia and Dario by repeating what already suggested by Dafne. Martina then argues: "If Giorgia goes backwards and Dario goes forward, the lines should do… hm… they should do (*crosses her arms*; Figure 6.35c) the exact opposites […] because suppose that Dario goes, Giorgia goes backwards and the line rises, while Dario goes forward, the line, his line goes down. Therefore, while one line is down the other line is higher".

(a) (b) (c) (d)

Figure 6.35. (a) Initial positions for Dario and Giorgia; (b) Dafne; (c) Martina; (d) the graphs

Giorgia and Dario, holding the controllers, move as expected and stop when they reach the ends of the tape strip: the graphs they created are shown in Figure 6.35d. Right after, Martina, quite satisfied, utters: “That’s what I meant, that the two lines crossed. And they also started from two different points”.

The experiment allows the children to verify both the aspects of their initial conjecture at the same time. It also creates the conditions for exploring the meaning of the crossing event, which emerges as a new singular event out of the problematic of new movements as a means to verify other than to produce something. The crossing of lines was further discussed in the following collective moment when the children investigated the reasons the event did happen (I do not focus on this other moment though, since it is not much significant to my discourse here).

Discussion

In the episode, the children put forward an experiment that allow them to understand whether their ‘double conjecture’ is true. But this verifying experiment, actually, does much more. In the choreography proposed by Dafne, and relaunched by Federico, the two children would have to move from opposite ends and in opposite directions. Federico adds that, in doing so, they cross each other, probably referring to the fact that, in swapping their positions, the two students will have to meet in space at some point. In choreographically imagining the experiment of Giorgia and Dario, Martina realises that also the lines “should do the exact opposites”. She does not directly refer the event of crossing to the lines, nor to the people, but, while speaking, crosses her arms for a moment as shown in Figure 6.35c. After the experiment, she is more than satisfied about the result since the graphs seem to match her expectations in terms of the overall configuration, which is now visible on the screen.

At this point of the teaching experiment, the children still have little experience with the software and, in general, about what the lines represent mathematically speaking. But this does not prevent them to grasp the potentiality of the two lines to cross each other.

Initially, the crossing event seems to be referred only to Giorgia and Dario, who will have to meet physically in space to swap their positions during the experiment. Martina instead seems to affirm that, in light of the initial conjecture and of the kinaesthetic experience of the two children, the lines should behave in an opposite manner. She also evokes the crossing of lines with her little gesture, without directly naming it.

One might argue that Martina's insight emerges from the logical continuity of movement (and lines): whether one line is continuous in going from up to down, while the other is moving in the opposite manner, there will necessarily be a point in which the two will meet, in the same way that the continuity of movement for the children is immediately related to their meeting in space. More importantly, I argue that the imagining of opposite lines emerges quite naturally from the thinking of lines and people that are in movement and, being in movement, generate new potential configurations that resemble and evocate each other.

This experiment also helps us understand how the crossing of lines is at the same time pivotal cognitively and generative ontologically. On the one hand, it does allow for connecting each single conjecture by means of a single experiment and for creating the conditions to interpret the graphs in terms of their mutual relationships, that is, in relational terms. On the other hand, it explodes the virtuality of the objects that populate the Cartesian plane: lines potentially cross each other and points emerge out of their potential crossing.

6.5.2 Luca and Luca's experiment

This episode starts with the collective discussion on the first day of the teaching experiment at lower secondary school. Thanks to the experiments of two students, Marco and Riccardo, the students have experienced that two straight lines can be obtained whether they stand still, and that their distance from the sensor influences the height of the lines on the Cartesian plane. The researcher then intends to discuss with the class a hypothesis that has been raised, by Marco and others, namely that of relating to and fro movements (facing the sensor) respectively to decreasing and increasing lines on the screen. While she speaks, two students in particular intervene with the urge of discussing other issues: one of them, Alberto, wants to verify whether the speed of the movements somehow does matter in WiiGraph; instead, Riccardo proposes "the matter of crossing", that is to verify "whether, when people meet, an X is formed, when people are parallel to, to the bar, an X is formed". I present here an excerpt of the discussion:

(Mar = Marco, Mat = Matteo, R = researcher, Ric = Riccardo, Sof = Sofia):

1. R: So, you do want to make an X

2. Ric: Yes, yes. Indeed, Alberto has asked whether he supposed that the X wasn't formed if people are parallel, if they are on the same line, to see whether they are or not

3. R: Hm… okay. Tell me, Matteo

4. Mat: In my opinion, it's not so true what he says, because if they're parallel, the lines are at the same distance so one should come from a bigger distance, the other from a smaller one and come back

5. R: Maybe, his thinking of parallel it's not the same you're thinking about

6. Mat: No, because if they're parallel the distance will always be the same, so we can't form an X

7. R: Ah

8. Mat: Unless, one goes backwards and the other...

9. R: But, in my opinion, when you say parallel you think of something different than when he says parallel… I think. I believe, I don't know, I think... say what you were saying

10. Ric: In my opinion, the test [the experiment] should be started when one person is behind and the other ahead and when, for example for me it's this line here of the tiles, they're on the same line, line of tiles for example, when they're at the same to, to the same line, on the same... they're... yes, on the same line, more or less

11. R: Can we refer to this (*points to the bar*) as a reference system?

12. Ric: Yes. Yes, yes, yes

13. R: When they are… how can we say that?

14. Ric: At the same tile… at the same distance from the bar

15. R: At the same distance

16. Mat: No!

17. R: So, what you were saying before about being parallel meant to be at the same distance from the bar

18. Ric: Yes, it means that if you're there, I'm next

19. R: Which isn't what you meant before when you thought of parallel? (*to Matteo*)

20. Mat: More importantly… if they were next to each other, the line we've seen it's always the same, it doesn't change, they don't meet each other

21. R: I didn't understand, sorry

22. Mat: If they're next to each other… if they're always next to each… even before we've seen that they never cross (inaudible)

23. Ric: Yes, but… wait a minute

24. Mar: They've started (follows Riccardo's argument)

25. Ric: If we start that one, if the test starts with one ahead and the other behind and they go backwards and then, when at some point they're next to each other, they stop

26.	Mat:	This is what I said before
27.	Ric:	A unique point is formed, and the line goes on. But if they go on, the X is formed
28.	R:	What do you want to say, Sofia?
29.	Sof:	I wanted to say that for me Riccardo is right. For me, if two people stay parallel… because, they are not doing the crossing, but they are parallel. The two rows, the two lines are one superimposed to the other, instead of doing a crossing, like when one goes backwards and the other forward. For me, they're a unique line, so to say
30.	R:	I don't know, I'm getting lost on this use of parallel, anyway...

Discussion

The use of the word "parallel" in the collective discussion made the students' discourse move across different perspectives in considering what is seen, what is done and what is imagined. In the discussion, Riccardo and Matteo initially seem to bring forth different ideas on how an X configuration might be created with two lines on the screen. Riccardo proposes that the experiment should help understand "Whether people meet, an X is formed, when people are parallel to, to the bar, an X is formed". Matteo disagrees, "because if they're parallel, the lines are at the same distance so one should come from a bigger distance, the other from a smaller one and come back" [10]. These students seem to contrast two opposite visions about the creation of an X-shaped configuration. Riccardo alternates a point-wise view on the event of crossing and a wider (comprehensive) vision of the same event, which is shared by Matteo. A point-wise vision considers the instant in which the distances of the two movers from the sensor are equal. Riccardo seems to suggest that, in that particular moment, the two people have to be parallel to each other, facing the sensor, with their feet on the same tile on the floor. A wider perspective on the event of crossing implies that, in order for the lines to meet each other, the two people have to move from opposite directions for creating a cross shape with both the lines. At the same time, these two complementary visions seem to be related to both a dynamic versus a static envisioning of what it means to be at the same distance. We might also interpret the alternating of perspectives in terms of the work of a choreographer, who alternates inside and outside view on a choreography, in order to appreciate the micro and macro structure of it. One might envision her body in movement, and another body shifting towards her, the wider event that culminates in the specific event of meeting in space and, at the same time, the crystallised position in which the two people stand at the same distance from the sensor and the corresponding lines overlap.

The term “parallel” is differently used in the discussion, in relation to both the lines and the people, and is productive of ambiguity that unfolds new horizons in the ways in which the experiment is imagined.

At this point, the students agree on making an experiment in which they have to create two lines that form an “X” (the researcher once again crosses her arms to stress the relative positions of the lines). Two students, Luca S. and Luca T., volunteer to take part in the experiment. At the request of the researcher to know how they will move, one of the students answers: “I go forward and he goes backwards” and, keeping the remotes in hand, the two mimic their movements, simply swapping their position. Some classmate relaunches the question of speed by saying: “Try to move a little fast”, causing stir and agitation in the rest of the students. The two movers rapidly look at each other and Luca T. says: “You go faster and I will go slow”, while the researcher tries to calm the class down. The experiment can finally start: the two students leave from opposite locations of the interaction space, and repeatedly move back and forth at different speeds (each trying to keep costant speed in his walk), alternating their movements’ directions (see Figure 6.36), without seeing any graph on the screen, since the option that hides the two lines is active. At the end of the experiment, the reasearcher reveals the two graphs that have been created (Figure 6.37).

The request of the initial task is, in some sense, achieved through the students’ movement, as the screen displays more than one “X” when the graphs are shown. The researcher astonished asks: “But didn’t we have to do something else?”. Riccardo exclaims: “Just one X? It’s impossible!”. At the same time Luca and Luca, who are in the interaction space, restart moving (without using the software): they exaggerate the incredible slowness of their new movement by slightly bending forward their back, raising their legs and stepping at a snail’s space, before the legs fall back down with all their body under its weight.

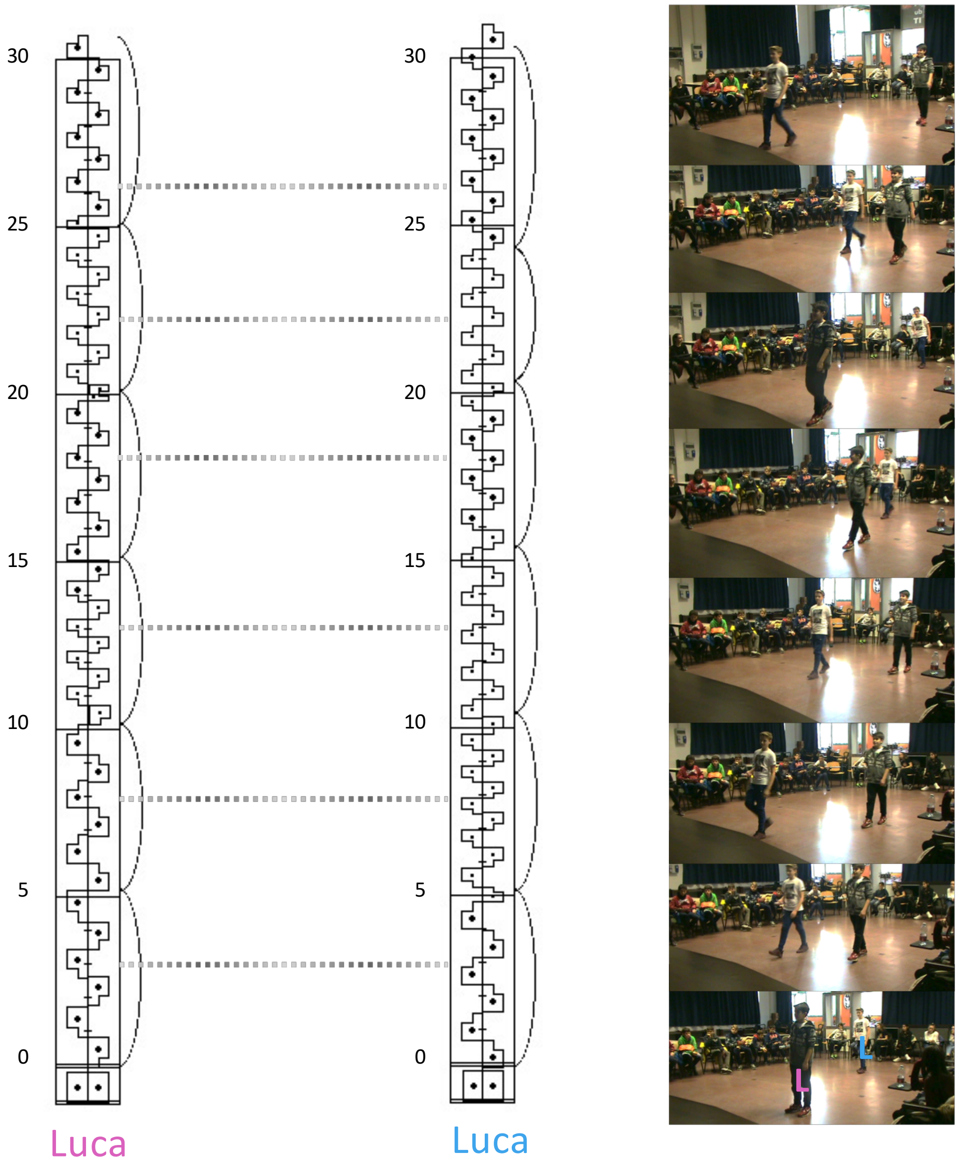


Figure 6.36. Movement notation for the experiment of Luca S. and Luca T.

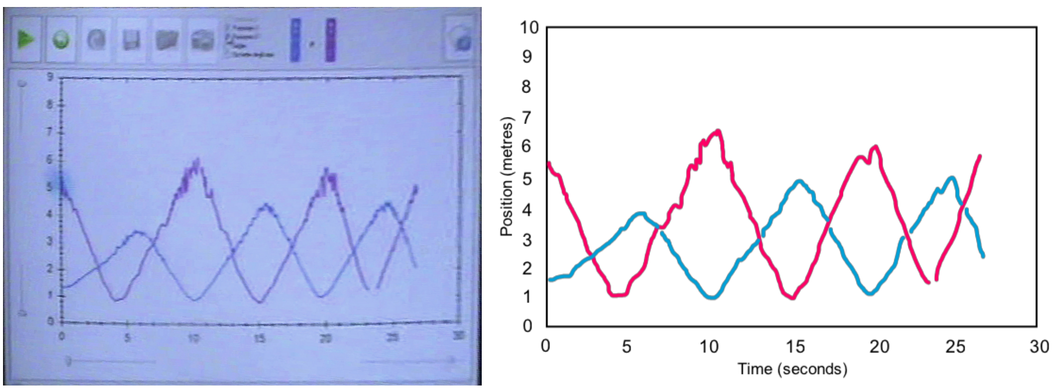


Figure 6.37. Luca and Luca's graphs

Discussion

Differently from the notations of the exploratory events I have presented in §6.3, the notation of this new experiment shows a precise structure, a sort of pattern, which is entangled with the request of the task. The students rhythmically go in one and the opposite direction, with a quite homogenous pace during each to and fro movement, emphasized by the round bows on the right of each staff, which also capture the changes in direction. In Chapter 5, I have discussed how the difference in speed inside the notation can be grasped by looking at the difference in length between symbols, since each symbol's length signals its duration. Since there is no clear indication about the length of the steps, this is not a matter of overall speed, but rather of more or less rapidity in stepping the space (forward or backwards). In particular, for this experience, infused with the rhythmic continuous back-and-forth moves, we can explicitly observe from the notation that (1) one student moves considerably faster than the other (right staff), and (2) the students change direction of movement more or less at the same time, no matter their pace (in fact, as a result, the faster one covers more space than the other; see Figure 6.37). They move to and fro at different speeds, but they almost seem to 'do the same thing' in opposite directions, as it emerges both from the video and from the notation. I see how the intervention of the students in the previous discussion implicates patterns of interference in what the class said about the possibility of creating a "X" on the screen: the challenge of exploring different speeds is tackled by Luca and Luca, who first agree about their ways of moving and then are caught in the repetitive and alternate movement, to which an outside observer (like myself) also surrenders.

It is interesting to point out how there are different intensities that populate and sustain the experiment: the students' urge of testing the relevance of speed, the preserved-in-movement relationship of the two students even without visual reference on the screen (no graphs are visible), the continuous repeated back-and-forth movement (three times), which creates a sense of an enjoyable routine to be treasured in movement. This is not a blind alignment, but rather a productive assembling of heterogeneous agencies and powers, which involve the individual movements, the trans-individual forces that percolate the classroom, the obligation to the task assigned by the researcher, and also the pre-individual level, which is exposed by the repetition and the implicit coordination of movements. In this case, the creation of lines that cross each other seems to be shaded by the relationality entailed in the event, and in fact the encounter of the students in the interaction space happens repetitively (even 6 times). What probably matters the most for the students moving, indeed, is not the overall shape of the graphs (the "X" shape) but the overall, broad feeling that the relationship between movements is continuously renewed.

The new experiment that later arises without being tasked explicitly has a completely different qualitative nature. The two students now seem to glide towards each other in slow motion, in an effort of reducing speed while emphasizing change. This happens quite naturally and again seems to be spurred more by a pre-conscious attunement to the nature of movement than by an analytical reflection on the graphical notation, shedding light on what is non-told but relevant in mathematical understanding, when we turn to thinking in movement.

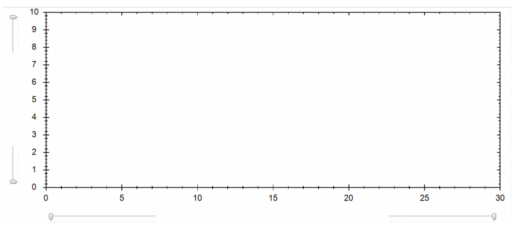
This episode also poses attention to how each motion experiment in some sense stands on its own but is not at all independent from the context or the agencies that populate the classroom interactions, which reverberate inside the mathematical representations by means of communicative, expressive and affective tones of movement.

6.5.3 Crossing hands

In this last section I focus on a segment of interaction between a group of students and myself while I was filming them, towards the end of a group activity in which the crossing of lines was discussed. First, I contextualise the moment and, then, I propose a discussion of the selected excerpt.

During the first day of the intervention, after the exploratory experiment described in §6.3.3 and the subsequent discussion and experiments, the upper secondary school students were divided into groups to work on a written worksheet (Scheda 1), whose tasks were as follows:

1. Think of two (non-identical) graphs and draw them below:



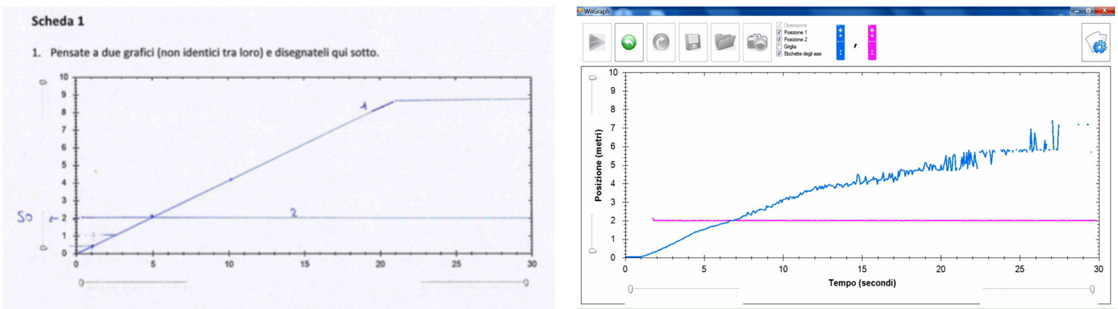
Describe with words the two movements that, in your opinion, provide these graphs with WiiGraph.

2. You have no more than two experiments with WiiGraph to try to produce the graphs that you drew. Explain how you would modify the movements' description after these experiments.

The group I was filming is composed by three students: Alberto, Arianna and Maddalena (Figure 6.38). They first drew two graphs on the given Cartesian plane, exactly the configuration with intersecting straight lines shown in Figure 6.39a. Then, they could move in front of the sensor to create their couple of graphs using the remotes.

Figure 6.38. From left to right: Alberto, Arianna and Maddalena

After the first session, the students were quite satisfied with the graphs they were able to produce (Figure 6.39b): the pink line was obtained by leaving the corresponding remote on a chair located two metres away from the sensor. Answering to the second question, the group brought forth the need for additional measuring tools (like a metre stick or a stop watch), which would have allowed better adherence to the drawn graphs, if used for marking the space to be covered and the time intervals in which a specific distance should be travelled.



(a) (b)

Figure 6.39. (a) The graphs drawn by the students; (b) the graphs created with WiiGraph

Since the students completed the task faster than the other groups, the researcher, who usually did not intervene in group work, came to interact with them in the left time:

(Al = Alberto, Ar = Arianna, Ma = Maddalena, R = researcher)

1. R: If we imagine that there are two people who hold the remotes and produce those two graphs there, exactly those two, is there a moment, or… anything, which the two lines have in common?
2. Ar: Yes
3. R: And the two movements?
4. Ar: There's the intersection point between the straight lines (*repetitively points to the intersection on the drawn configuration*) that represents the moment when both people are at the same distance (Figure 6.40a)
5. R: Hm. Therefore, what does it happen in that point?
6. Ar: It means that, while an object is stationary, the other straight line, which represents a speed, begins increasing (Figure 6.40b) to the point where it travels the space at which the second straight line is (*positions the pen on the worksheet, overlapping the horizontal straight line*)
7. R: And then?
8. Al: Yes, substantially
9. Ar: And then it overtakes it. It begins covering more space
10. Al: Yes, the two persons are at the same distance in the same time
11. Ar: After which the one who has a speed covers more space 'cause that has to continue to always travel the same space over the flowing time
12. Al: But we had, maybe, instead of going on straight here, we had come down to zero again, we would have had two intersection points, in two different moments, 'cause
13. Ar: That went back to travel, to the space of two metres, like
14. Al: There would have been another point in which, following their motion, the two straight lines would have met at the same distance at the same time
15. R: The straight lines or the people?
16. Al, Ma: The two people! (*laugh*)
17. Ar: Just imagine that if a person is at this distance from the stationary, the other begins travelling through space, they meet and then go on, if they come back again (Figure 6.40c).

(a)

Keeps the hands as in figure, emphasizes the relative positions by slightly moving the hands back and forth three/four times, then moves the hands towards each other

(b)

Slowly moves upward the right arm, while the left arm is on the table with left hand kept straighten and steady (overlapped images with transparency filter)

(c)

Choreographic movement for the crossing event: Arianna keeps the left hand steady while moving her right hand from far towards her torso, then decelerates to mark the crossing position, overcomes it, and then goes back to the initial position. Alberto follows her, but repeats the choreography two times, with his own pace

Figure 6.40. The group gesturing about the crossing of lines

Discussion

The three students focus on the event of crossing lines through the explicit request of the researcher and, in the brief excerpt, we see the unfolding of meanings that actualises the potential mobility of the chosen configuration. The episode helps us recover what already presented above about the overall theme of the section. Arianna and Alberto both interpret the point of intersection as “the moment when both people are at the same distance” [4] and the point in which “two persons are at the same distance in the same time”, that is, in terms of the modelling situation. Moreover, the particular configuration that they drew refers to the event of crossing as that in which one overtakes the other because of her higher (non-null) speed, and the continuous flowing of time [9, 10]. Alberto also brings into life a new potential intersection, which arises from perturbing the initial situation. Imagining modifying the last part of the line, a new intersection emerges out of the new configuration. Again, this allows to tap into the ontological work of the students, for which the lines are potentially open to intersections and perturbations. Maddalena silently partakes in the interactions but joins the laughs of her group-mates when the provoking question of the researcher [15] catches her into discourse.

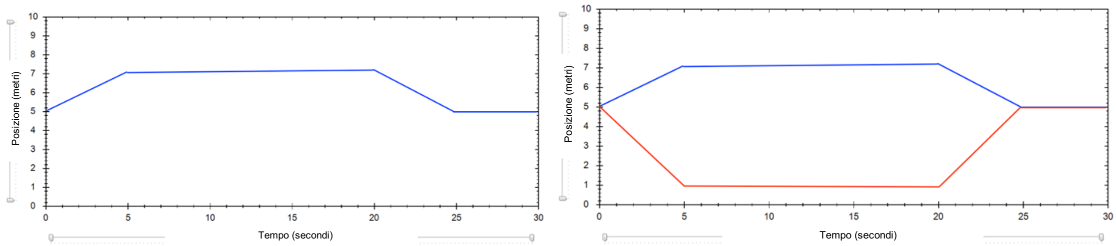
As a last concern, the choreography for the crossing event, which closes the episode, hatches the imagined qualitative nature of the crossing event. In relating the two lines and the two people (indifferently), the two hands embody the relational movement that is actualised in gestures. Alberto and Arianna move differently to perform the choreography but stay parallel to each other and their movements are controlled and focussed, with their hands kept straightened facing their own body. Arianna emphasizes the intersection by slowing down her movement when the right hand reaches the left hand and gazing at the camera. Alberto instead repeats the choreography two times, with faster shifts than Arianna's and moves in a sustained manner, intently looking at his hands.

6.6 Speed of movements, speed of lines

This section focuses on episodes that are somehow related to the concept of speed, which is implicated in the particular representations in WiiGraph. Speed can be seen as a relational concept, one that emerges from the coordination of space and time. In the context of experimenting with WiiGraph, speed is both perceived, or felt, when I am moving, and seen, when I am looking at someone moving or at the screen when one or two (and even more) lines are originating in real time. Time (the modelling time), which is the same for each remote, guarantees to the expert user that the graphs are progressively produced at each and the same instant. Nevertheless, we have observed that, at some point, the students questioned on the speed of the lines, and especially on whether the two (or more) lines move at the same speed or not. This is a very rich and intriguing aspect of encountering and examining graphical representations of spatio-temporal functional relationships. If we take time as a parameter, we have no doubt in affirming that the speed at which the lines move (originate) on the screen is the same for all the lines. If a graph, though, is a moving line, it can – in principle – possess each and every degree of freedom, and proper features, even proper speed. In this section, I want to investigate this issue, in particular how lower secondary school students make sense of speed. One point of the discourse will even touch on differences and relationships between the speed of movements and what I have called the 'speed of lines'.

6.6.1 Rob & Bob

The activity of Rob and Bob consisted of a written worksheet (Scheda 3) assigned to the higher and lower secondary school students involved in the study, with slightly modified versions. A similar version of this task had been proposed in a previous study and some insights are discussed in Ferrara & Ferrari (2017b). The worksheet presents to the students Rob and Bob, two little robots, which are told able to move in a very precise way. Also, the little robots have to be imagined using WiiGraph to make a 30-second experiment. Against Rob's movement, WiiGraph produces the line depicted in Figure 6.41a.



(a) (b)

Figure 6.41. (a) Rob's graph in Scheda 3 and (b) expected solution with Bob's graph added

The task given to the lower secondary school students then asks:

> Bob also moved, but its line is hidden!
> We only know that Bob started together with Rob, at the same distance from the sensor, but always moved at a double speed and in opposite direction.
>
> - In your opinion, which line would WiiGraph show for Bob's movement?
> - Once started, did Rob and Bob meet again?
>
> Explain your reasoning.

The students worked in groups to solve the task. A collective discussion then allowed for comparing different solutions from the different groups. The task has an unconventional nature with respect to the mere shift from model to motion or vice versa. Indeed, information about the missing graph is given in terms of the relationships between the two robots' movements ("double speed", "opposite direction"), so that the students are moved to think about the relationships between the two graphs (double slope with opposite sign), through their perceptual and bodily experience with the tool. In addition, the simultaneity of the two movements, which is legitimate by the peculiar way of producing representations by the software, is embedded in information about the starting instant and Bob's position ("Bob started together with Rob", "at the same distance").

The given graph is that of a piecewise function made up of four pieces, which capture alternate ways of moving by Rob: walking further from the sensor for the first five seconds, stopping for the next fifteen seconds, returning to the starting position in other five seconds, and stopping for the last five seconds (Rob keeps constant speed in each time segment). The expected solution is the red graph in Figure 6.41b. It is the graph of a piecewise function again made up of four pieces, each defined on the same sequence of time segments of the given graph. These pieces correspond to four ways of moving by Bob: getting close to the sensor for the first five seconds, stopping for the next fifteen seconds, returning to the starting position in other five seconds, and stopping for the last five seconds. However, Bob is supposed to cover double space with respect to Rob, according to the constraint of moving at a double speed (naturally, this is true when he moves, and trivially also when he does not, since the covered distance is null).

I want to focus on different unexpected solutions from the analysis of groups' diagrams, dwelling on their potential to bring forth new relational possibilities for the two robots' movements as well as for the pair of lines. In the following, I take these solutions as the problematic actualisations of the mathematical events that the groups encounter in solving the task. The prototypical solutions given by the students are offered in Figure 6.42 (a to d).

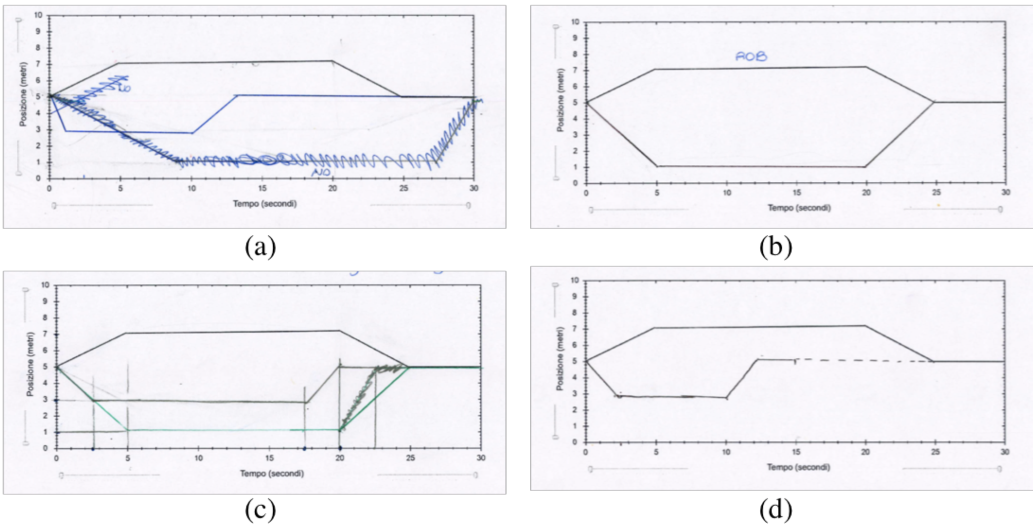


Figure 6.42. Prototypical solutions for the written task of Scheda 2

All the diagrams take into account the information about the opposite direction of Bob: the capture of this constraint in the diagrammatic is not an issue for the students. Moreover, each added graph is made up of four pieces, reflecting the opposite way of moving of Bob with respect to Rob: first getting close, then returning to the start (first a decreasing piece, then an increasing piece). Not even slope seems to be a difficult issue to tackle in this context: the double speed of movement is given as double slope for each 'slanted' trait in three out of the four diagrams (in Figure 6.42a, the initial slope seems to be five times the given one). However, the duration of Bob's movement is problematic for the students. In fact, while in all solutions there is correspondence between ways of moving, not all of them capture the embodiment of duration. In the graphs a, c and d of Figure 6.42 there are time segments in which one robot should be moving, while the other one is standing still. This does mean that the constraint for Bob to move *always* at a double speed with respect to Rob is not preserved. Thus, the problematics of duration intervene in this task in unexpected ways. These problematics seem to break with causal connections and direct determination, opening up to speculative and inventive investments and to a generative movement, implicating the perturbation of spatio-temporal relationships. In the graphs a, c and d of Figure 6.42, thinking in terms of the modelling situation, Bob already stands still while Rob is still moving and, later, Bob moves towards the starting position while Rob is still standing still. In the graph of Figure 6.42d the second robot stops right after fifteen seconds, in the very middle of the experience, and ideally disappears from the view of the sensor.

The collective discussion after group work reveals that some of the groups engaged with the task kinaesthetically envisioning the event of the robots' experiment as one in which they were covering the same amount of space, the same *route*.

In principle there is logical equivalence between considering "double speed" as actualised in a double distance travelled within the same time segment and the same distance travelled in half of the time. Whether there is an already given route to be travelled for each of the actors, the problematic of covering fixed space drives, in the diagramming of the missing graph, students' perception and visualisation to considering this distance as necessarily covered in shorter time by the faster actor. The graph in Figure 6.42d is the most coherent in respect to the axiomatic way of reasoning about double speed (the graph 'lasts' only 15 seconds) but at the same time is the one that departs the more from the kinaesthetic experience with the technology. Briefly speaking, it is nothing but a temporal shrinking of the given graph. But achieving that graph would mean disappearing completely (and instantaneously) from the sensor's view. Instead, the other graphs embrace all the thirty seconds of the session. For example, in Figure 6.42a the graphical representation is almost the same of Figure 6.42d, but it is prolonged with a horizontal segment until the 30-second right limit. The same happens within the graph in Figure 6.42c, which is particular though, since it struggles to depict the simultaneity of the two robots' movements, by considering the same amount of time for the intermediate 'stop', but each of the slanted parts is travelled in half time.

The collective discussion unfolds the event-nature of unexpected threads traversed in solving the task. We see how the students inscribe themselves into the temporality of imaginary situations with the robots. For some students, the constraint of double speed implies that Bob only moved for 15 seconds. For example, Matteo stresses:

> Matteo: It says that he took half the time [...] twice as much the speed. Therefore, it means that he stopped at 15 [seconds], and then he [Rob] stood still. [...] If he went at a double speed, then he had to stay still for a while. Because he cannot take up the same time of Bob, he should do the route two times. [...] If Rob takes 30 seconds and does that route there, Bob at 15 seconds finishes that route

Gianluca agrees with Matteo and explains:

> Gianluca: We did that Bob did two routes with respect to what Rob did

The collective discussion helps these students bring to the surface taken for granted assumption about the experiment (like the fixed route) and the contradictions within some parts of their diagrams (when one robot moves, the other also has to move).

The ways of perceiving temporality are different for different (groups of) students: inscribing oneself into the experiment is to enter the event. In this particular event, time is actualised as both duration and simultaneity of movements. These aspects, entangled with qualitative dimensions of movement, like speed, become problematic for learners and generative of new ways of navigating the graphs, which are crucial in making sense with WiiGraph of time as the independent variable in the constitution of a configuration.

In closing the discussion of this episode, I would like to stress the power of disrupting activities like that of Rob and Bob for classroom practice. By challenging common conceptions of space and time, we are actually asking the students to 'enter the graph' with their own sensitivities to movement, and to explore the situation by means of kinaesthetic imaginations of the event. The collective discussion brings together these different ways of inhabiting the graph and brings to the surface those elements, like the fixed route, that are subjectively added to the task. The episode also sheds some light on the role of temporality in making sense of speed in spatio-temporal representations.

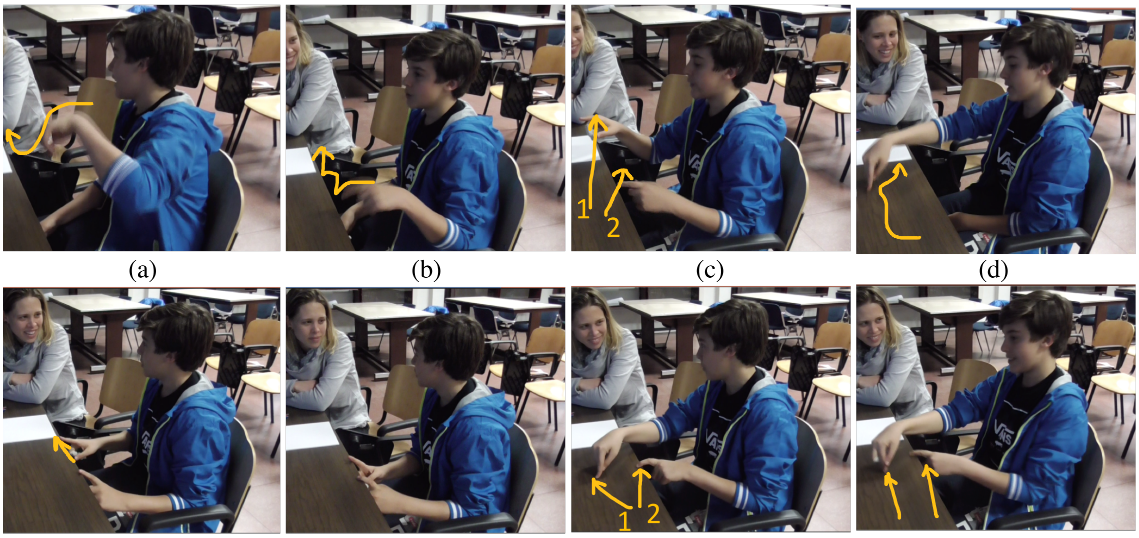
6.6.2 Gianluca's interview

This episode presents a brief excerpt of Gianluca's interview at the end of the teaching experiment. In the 2-minute segment, at the very beginning of the interview, Gianluca recalls the activity of Rob and Bob, which I have just discussed in the previous subsection, as a way of bringing forth his personal entanglement with the intervention.

(G = Gianluca, R = researcher)

1. R: I meant to ask you… if there's something particular about this work that you liked, that struck you, that…
2. G: It struck me that anyhow, during the work, I've learnt to explain, that is, how important explanations are… 'cause, I dunno, perhaps I answered no and that already seemed to me the explanation, namely, I said no and therefore no (*shrugs his shoulder and smiles*). And that's when I realised that we have to explain, 'cause the one who reads the questions perhaps never tried to answer the problem, if she's not able to solve it and reads the explanation perhaps this helps with it. So, I've understood how important it is to explain the reasoning I did.
3. R: When you say this… are you thinking of a particular problem, among those we did together?
4. G: Well, for example Rob and Bob, that about speed (Figure 6.43a), where the one moved at a double speed with respect to the other, a-and I'd thought that, hm, Bob, which went at a double speed, hm, did two times the same route (Figure 6.43b), but in fact, then, with the others' explanation I've been able to understand that he actually had to keep the same speed. That is, double speed, when, say, if Rob moved, he [Bob] had to move at a double speed (Figure 6.43c), if he [Rob] was standing there (*gestures a piece of horizontal straight line*) the double speed of zero is zero and, therefore, he [Bob] also had to stand there (*again gestures a piece of horizontal straight line*). So, thanks to the explanation made by another, I was able to understand my mistake and modify it
5. R: Interesting. Concerning Rob and Bob, why did you think they did the route twice?
6. G: That is, 'cause if Rob did a route (Figure 6.43d), hm, I'd reasoned on the entire duration (Figure 6.43e). Namely, if Rob had done that route in 30 seconds, I thought Bob will do the same route but in 15 seconds (Figure 6.43f),

> hm… And that's when I realised I'd to, when the one moved the other had to move (Figure 6.43g), because he had to keep double speed (Figure 6.43h), therefore I realised my mistake.



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 6.43. Gianluca's gestures during his interview

Discussion

The initial very general question of the researcher mainly aimed at creating a sufficiently informal atmosphere in the interview. In his answer, Gianluca does not bring to the fore a playful moment or a fun experience like other interviewees did, rather he focuses on the importance played by explanations. In so doing, he claims that explanations have been crucial in the teaching experiment in more than one direction. He claims that *he* had learnt this. Explanations have become meaningful for him in relation to mathematical problems, since they can constitute a valid help in answering and solving the problems. He expresses a change in his way of considering explanations as more than just monolithic answers to a particular task. When the researcher (positively surprised by the answer) asks whether he is associating this issue with a specific problem of the teaching experiment, Gianluca reveals that the problem in question regards speed and the task of Rob and Bob. He also 'confesses' that, in that context, he benefitted from explanations made by others to understand his own mistake, particularly the fact that the slower robot could not have covered twice as much the space travelled from the other robot. The meta-cognitive level of Gianluca's thinking emerges from rethinking the past situation that was positively resolved with the help of others. In few but complex and articulated utterances, he condenses his initial thoughts about the task and the elements of the resolution, which were crucial to understand the diagram against Bob's movement. On the one side, Bob's doubling of the route. Once Rob and Bob are introduced in speech, Gianluca gesticulates a lot, even in a nervous way, actualising the lines associated to the two robots with both his hands moving first in front, and then on the table. The moving hands distinguish the path of the two lines, one after the other, when necessary to bring forth the moments in which the robots make different things (the slanted parts) but both have to move (Figures 6.43c first and then 6.43g): this excludes the possibility, for Bob, of covering twice the same route. On the other side, the issue of null speeds and their relationship. The hands shift together when reference is to stationary moments (Figure 6.43h), to convey that the relationship between the two movements is always on, even when the robots stop, therefore it is never the case that each of the robots does move whether the other does not. Coordination is found in the second case, lost, and unnecessary, in the first case, where separation is the way of coordinating the two lines, and movements. What interestingly emerges from the movements of the hands, therefore, is the need for a relational vision of the robots' speeds (and the configuration) to successfully understand the situation.

6.6.3 The fish becoming whale

During the fourth day, the senior researcher (Francesca) and Giulia (the master student who participated in the intervention as observer) moved in front of the sensor to do an experiment with the remotes. They looked for coordination in movement: starting from the same point in space, they moved quite at the same speed but in opposite directions, then stood still for the same seconds and finally moved again, simultaneously, trying to keep same speed once more, and swapping their positions. The software recorded these movements throughout the experiment, but the two lines were hidden, and the students could not see the corresponding graphs. While Francesca and Giulia were moving, one student, Marco, used both his index fingers, coordinated with each other and with the event, to mime the two lines together, in an effort of reproducing the graphs as they would have been shown on the screen.

The students, sitting all around the interaction space, observed the movements and were tasked in pairs with predicting and drawing the graphs produced by the experiment with WiiGraph. Figure 6.44 shows three of the most commonly predicted configurations, which mainly differ from each other concerning horizontal pieces. Francesca and Giulia's coordinated movements actually implicate a symmetric configuration: two graphs that, when put together on the same Cartesian plane, create a shape similar to the stylised picture of a fish (see Figure 6.45a).

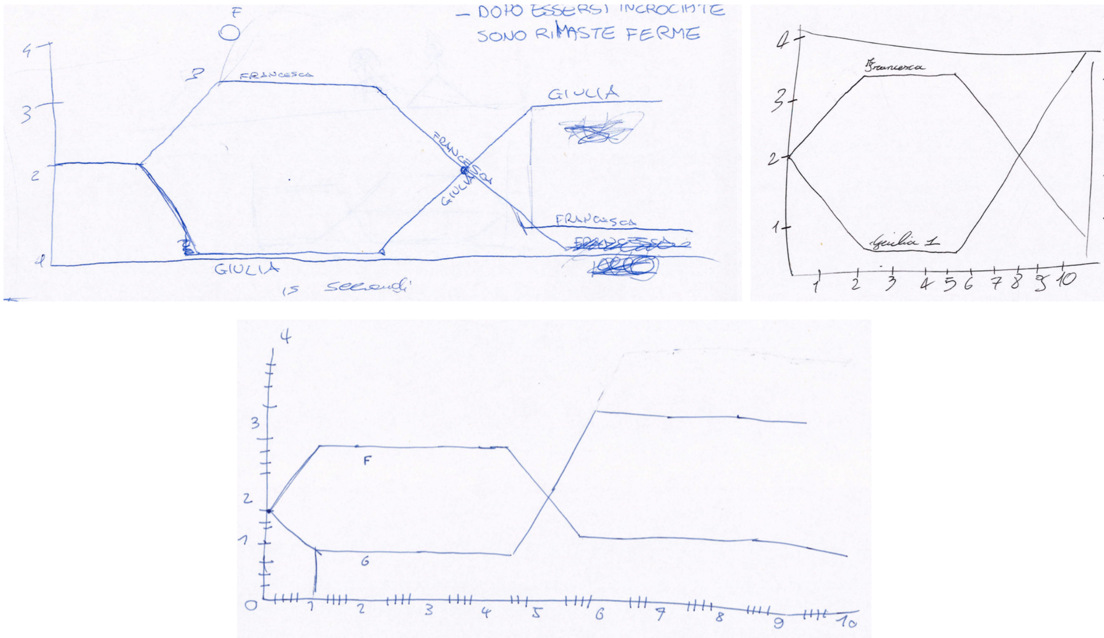
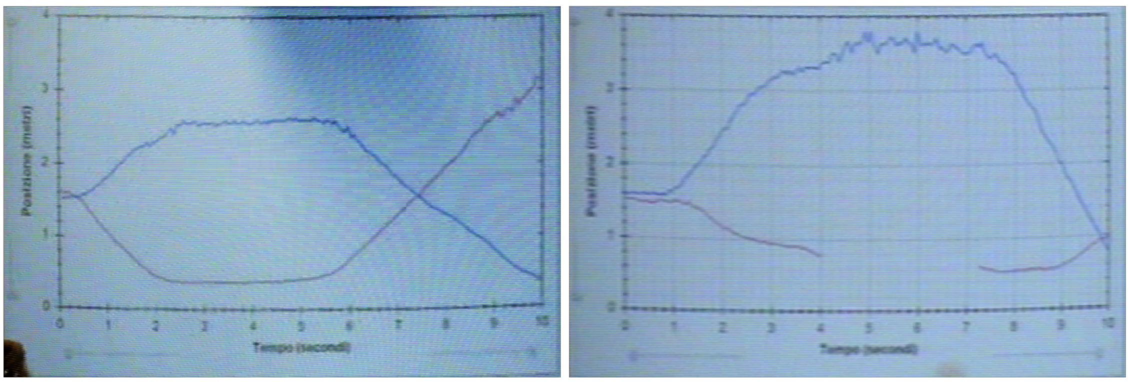


Figure 6.44. Graphical predictions drawn by three pairs of students

We go back to the moment right after the task, when all the expected configurations are shared in a classroom discussion, and the lines created with the software are finally revealed (Figure 6.45a again). It is at this point that the students aloud claim that the configuration has a fish-like shape. I am going to focus on a moment of the discussion in which the students create a new configuration, which resembles more a "whale" than a fish (Figure 6.45b).



(a) (b)

Figure 6.45. (a) The fish becoming (b) whale

The collective discussion develops through detailed observations and measurements on and around the configuration. The students first focus on the intercepts, and on the horizontal traits of the graphs. Then, one student, Matteo, asserts that Francesca (blue remote) has moved faster than Giulia (pink remote). Another student, Gianluca, draws attention to the initial part of the graphs, for which he affirms that "they covered the same metres in the same time", while, looking at the final part and comparing the pieces composing the tail of the fish, he rather argues that Giulia moved faster. This claim makes space for discussing the (a)symmetry of the configuration, and the role played by speed in unfolding the (a)symmetric relationship between the two lines:

Alberto:	Does the speed of the steps matter? In my opinion, yes
Davide:	Giulia rose faster, in fact [it] arrives higher
Sofia:	Giulia moved faster, but because she also travelled more space, therefore she had to catch up with Francesca, therefore doing more space, she went faster

At some point, the researcher plays the built-in replay of the software so that the students can see the previous graphs originating in real time. Watching the replay, someone argues that one line arrives earlier at the end of the graphical window. Others say that this is only perceived because the one rises faster. The researcher therefore suggests making a new experiment in order to check whether speed matters, while asking to keep the goal of creating a fish-like configuration (imagining to direct main attention to the issue of symmetry).

Alberto and Davide volunteer for this experiment: they share instructions and decisions about the types of movements to perform (Alberto has to move faster, they have to start from the same position, etc.). Then, the experiment takes place and, surprisingly, the two students create the graphs of Figure 6.45b (Alberto: blue line, Davide: pink line). At the end, the rest of the class giggles and laughs:

(A = Alberto, D = Davide, G = Gianluca, M = Matteo, R = researcher, S = Sofia)

1.	R:	Rather than a fish, it looks like… well
2.	G:	A whale!

Lot of laughs from the class; others say: "a whale" or "an obese fish", "an adult catfish"

3.	R:	Can you tell us… (*addresses Alberto and Davide*)
4.	D:	Well, what we can…
5.	R:	… with respect to what we were verifying?

6.	A:	Anyway, the speed doesn't matter. The speed does not matter
7.	G:	But yes, it does change. Look, here (*points to the first part of the pink line;* Figure 6.45b), he [Davide] was going slowly, and he get here and (*inaudible*)
8.	D:	Can I say something?
9.	A:	That the speed doesn't matter
10.	S:	The speed of the steps! (*points to Alberto*)
11.	D:	He [Alberto] was going faster and in fact the line…
12.	G:	Yes, it matters
13.	M:	Since he [Alberto] was faster, he surely covered more distance
14.	G:	Faster (*almost inaudible*)
15.	D:	In the same time [*segment*] I barely did half a metre, while in the same time he [Alberto] did two
16.	G:	Exactly
17.	R:	Does the speed matter or doesn't?
18.	G:	It matters, of course it matters (*Davide echoes him*)
19.	R:	Because Sofia said that the speed of the steps doesn't matter
20.	S:	Yes, well, I was saying that you cannot see, well, in the graph it can't be seen that he [Alberto] goes faster, because the lines go in the same seconds, but if
21.	R:	So, you say, if I start the replay…
22.	S:	It matters to change, the speed doesn't matter, but it matters to change the shape, the path, well, I mean, you cannot see the speed in the graph, but you can see the difference between the two, in the path, but you can't see the speed, you can only see that he [Alberto] covered more way
23.	A:	Yes, it matters in the sense that
24.	M:	In the last part there, it gets bigger
25.	S:	Yes, but you cannot see that he [Alberto] goes faster

Discussion

In the episode, we see how the students struggle with the concept of speed as it irresistibly emerges out of the fish first and then of the whale. In fact, the situation plays out different speeds: the speed of the steps (related to people and their movement, which I also refer to as speed of movements), the speed of the lines (related to the movement of the lines while they are created). The question of whether or not speed matters, as well as whether or not speed is visible in the graphs, is a problematic one for the students. In which sense, indeed, we ask whether speed matters? Movement is everywhere. Watching the replay, one line is perceived as (moving) faster than the other. Lines move, precisely they move following the bodies in movement with the remotes in the interaction space but are also moved by the modelling time in the graphical space. Therefore, lines can be told and seen as arriving "higher" and going "faster" than one another, but also both going "in the same seconds". Here is most probably where and why the speed of lines and the speed of movements are (con)fused, each infused with the quality of the other. Therefore, some students think of the lines rather than the people as moving at different speeds. The new configuration resembling a whale breaks the quasi-symmetry of the fish, opening up new possibilities that keep unaltered some relationships while changing others. Alberto and Sofia seem to think that speed does not matter, at least the speed of the steps, while others, especially Gianluca, are convinced that speed does matter. Everything is relational. The students speak and think of speed by relating it to the movements, and vice versa they speak and think of movement by relating it to the speeds, no matter whether of the steps or of the lines. The students speak of one line in relation to the other, of one movement in relation to the other. The fish becoming whale is a metaphor through which capturing this relationality: the fish becomes a whale through changes in the relationship between the speeds of the two movements, even if directions and 'rules' are preserved. The whale is a qualitative alteration of the fish, one in a multiplicity, in the virtuality that speaks directly to the (a)symmetry of the fish-like configuration. In this sense, the fish and the whale live on a *continuum*, which relies on the micro-perceptions and the different points of view at play within the classroom and captures the potentiality of movement in the given configuration.

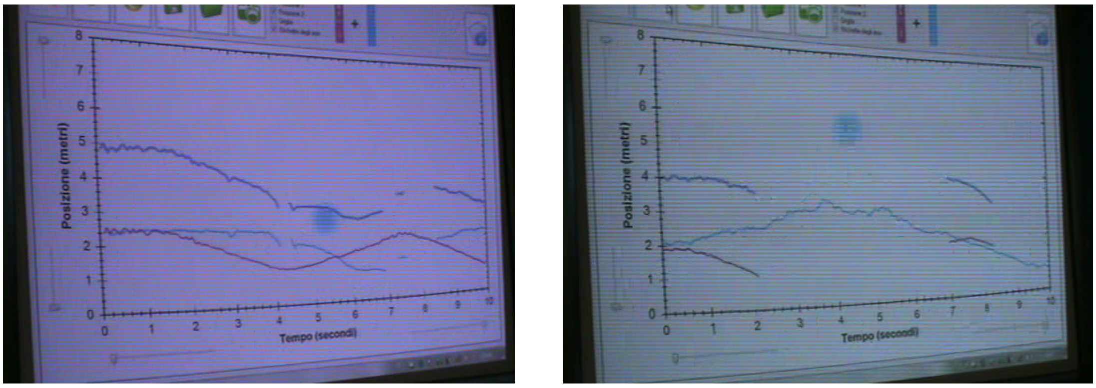
6.7 Sum graphs

This section focuses on the experiences of the upper secondary school students with sum graphs. In the Introduction, I have already mentioned the functioning of this modality of WiiGraph and possible implications for classroom activities. Briefly, the software shows on the screen the two position versus time graphs (pink and blue lines) together with an additional graph (dark blue line) which provides the sum of the other two positions instant by instant. The aim is here to recover and expand some of the considerations concerning the present study, for which the class of grade 10 students also worked on sum graphs. During the third day of the teaching experiment, the students first explored the three graphs without being told what the third line was, with the aim of discovering it. They investigated the sum by doing experiments in a collective discussion led by the senior researcher. In the following day, they also faced a written task (Scheda 5) working in groups, and after a new collective discussion were tasked with an individual worksheet (Scheda 6). At the beginning of the last day, the individual written activity was discussed within the classroom, showing some of the students' solutions and the software was used to test conjectures.

In the following subsections I focus on the exploration of sum graphs and on the graphical productions shared in the last collective discussion, as a means of examining the unfolding of meanings for the sum in this particular context.

6.7.1 Exploring sum graph

During day 3, the senior researcher proposes the students the exploration of a new modality. Arianna and Giulia move freely and, with a first exploratory experiment, produce the graphs shown in Figure 6.46a:



(a) (b)

Figure 6.46. (a) First and (b) second exploratory experiment with the sum in WiiGraph

While the two students are moving, a general sense of astonishment and bewilderment permeate the class, and questions arise from the sitting students, like: "Which is the third remote?", "Is that the average?". A second experiment is carried out to further explore the new situation, with considerable technical issues so that the pink and dark blue line disappear from the screen much of the time (Figure 6.46b). A first hypothesis is shared in the class, according to which the third line could be the average of the other two lines, as Annalia suggests:

Annalia: There's no third remote, therefore I suppose that the third line is almost a union of the other two and the union can't exactly be, then a union that can be between two movements is an average.

Following this first hypothesis, Alberto proposes that a suitable experiment to verify whether the third line is the arithmetic mean of the two distances is one in which the two girls stand still in a position: in fact, the average line would be in between the other two lines. Suddenly Maddalena interrupts the discussion by noticing that among the graphs on the screen the two (pink and blue) lines have the same starting point, therefore "if the initial point [of the third line] isn't the same, it can't be the average!", and adds: "it's not in between the two lines, so it can't be the average". Giuliana follows on her comment saying:

Giuliana: They started at two meters from the reference system. The theoretical average line starts from four, it's not an average

Discussion

In this first part of the episode we see the emergence of a first hypothesis in relation to the nature of the third line, which immediately connects this line and the average of the other two. In the study conducted for the master thesis, I also noticed this as the first conjecture regarding sum graphs in exploratory experiments with WiiGraph. Certainly, it cannot be generalized, but it is peculiar that just this idea emerges from sum graphs, since, algebraically, the average is the weighted sum of two quantities (whichever the quantities), that is, ultimately a particular kind of sum, divided by the number of elements that contribute to it.

It is also relevant for our discourse that the third line that appears on the screen made the students guess about the *absence* of a third remote, the one generating the third line. In the Introduction, we already discussed the thought experiment of one student, Alessandro, who imagined the *presence* of a third moving person to explain the constraint and functioning of the sum graph. In this new episode, we observe the negative determination of the third person, which implies a relationship between the other two elements that is generative of the third line. Annalia's comment, in fact, expresses the urge to compose the other two in specific ways, as she proposes to join them together with a kind of union.

This first hypothesis is then abandoned when focus is driven to a particular point of the graph, and the conjectured relationship between the three lines does not hold. This is explicit in the comments of Maddalena and Giuliana, who both refer to the intercepts of the lines to highlight the discrepancy between what would be expected (by the average) in theory and the given graph. Indeed, Maddalena brings forth the argument that the initial point of the average should be the same as the initial points of the other two graphs, since the pink and blue lines start from the same position. At the same time, borrowing from the experiment's proposal of Alberto, who suggests that, in the case of two horizontal straight lines, the average should stay in-between, she also points out that the third line cannot be the average since it does not possess this feature of being in the middle. Giuliana stresses a similar point and emphasizes the breaking down of the hypothesis when saying: "The theoretical average line starts from four, it's not an average".

Then, other students propose the sum as a solution but seem unconvinced, so the researcher suggests them to try the experiment proposed by Alberto. Arianna and Giulia stand still for all the duration of the experiment: their graphs are shown in Figure 6.47.

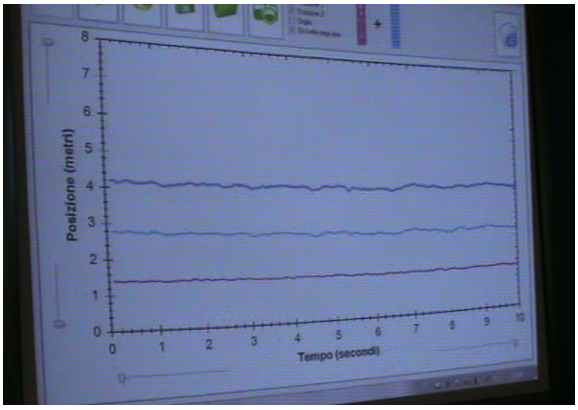


Figure 6.47. Sum graph (1)

After this experiment, in response to the researcher's question: "Is it the average?", the students relaunch with very diverse conjectures: the symmetry axis, the sum again (depending on time or distance), twice as much the position, the difference. They seem to definitely exclude the idea of the average, but this new configuration engenders new features that become relevant to discern the relationship between the graphs. For example, Giuliana intervenes again in the discussion:

Giuliana: The distance between blue and pink is the same as that between dark blue and blue

Discussion

The experiment proposed by Alberto draws on the feature of the average of being in-between, which would be straightforwardly realised by a dark blue line in the middle. It captures the average as the composition of two elements, which gives a third element spatially enclosed between the other two.

This way of thinking about the average, though, also implicates the difference between the two lines, so to speak, since *being in the middle* might be referred to the operation of halving the distance between the two lines, in order to find the average value. The operation of halving the distance would be, indeed, related to difference. For example, whether one line is at a height of 4, while the other is at 8, the average would be at 6, having the same distance (2) from the other two lines. The choice of straight lines as the most convenient one, even if it remains implicit, is also relevant in this context, since it reduces the complexity of a graph to a single value, which can be treated as a unique number all the way through. Unexpectedly, the experiment spurs fragmentary interventions of the students and opens up a constellation of possibilities that arise from the specific configuration. The experiment seems to open up new potential meanings for the third line than before, even though now the graphs are simple horizontal straight lines. Giuliana underlines that the distance between the blue line and each of the other two lines is preserved. Yet this emerges out of the specific configuration, since Arianna and Giulia stood (and still stand) at distances from the sensor, which are such that one is almost twice as much the other. Shifting attention to this aspect invokes the relationships to be spotted in-between again, but in a different way, namely looking at pairs of relationships that can be preserved.

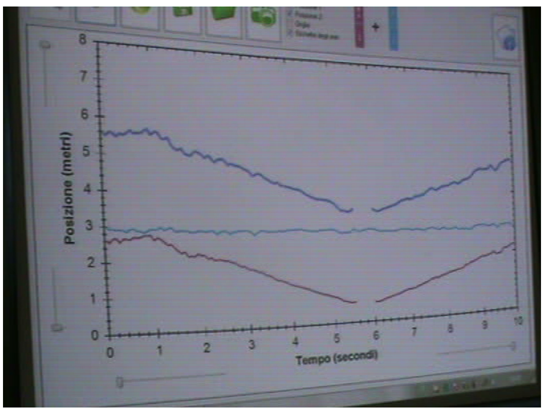


Figure 6.48. Sum graph (2)

A new experiment is proposed: one of the girls stands still, while the other one freely moves to and fro (the graphs that appear on the screen are shown in Figure 6.48).

After the experiment, a brief discussion unfolds as follows (R = researcher):

Silvia: It's the parallel line to Giulia's [graph: pink line] that, though, starts, as a starting point, from the sum of the starting point of the other two

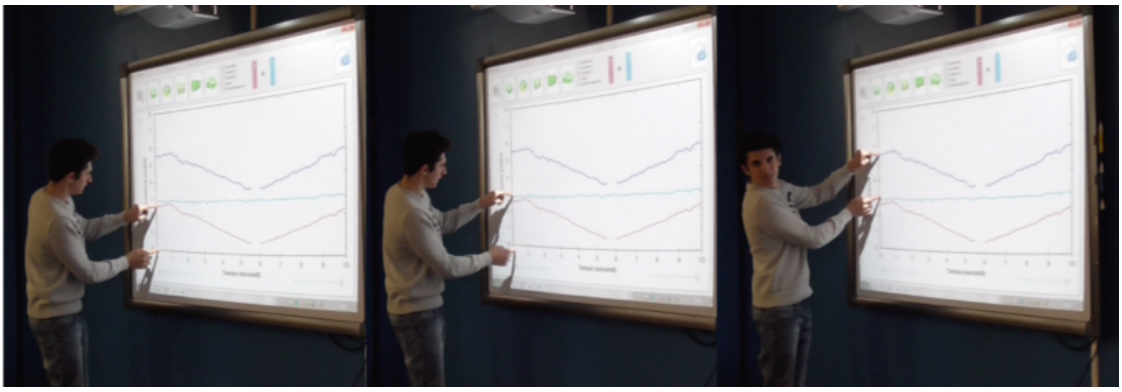
Giuliana: But, it's parallel only because the other is steady, hm, it's parallel only because the other one is steady (*mimes a straight line in front of her with her right hand*)

R: Why?

Giulia: But would it be still parallel to my line whether we both move? […]

Alberto: In my opinion, it's the sum of the distances from the sensor (*goes to the IWB*), this line remains constant (*points to the blue line*), this line remains constant, while this one, this pink (*follows with his right index finger the pink line*) changes. So, for example, at the moment, at six in which I have this point (Figure 6.49a), I also have it here (Figure 6.49b), here (Figure 6.49c), because it would be, this distance, plus this distance (*repeatedly moves his hands up and down along the vertical axis, from the blue line to the horizontal axis, and more or less in the same way while referring to the two distances*), so it's, tac (Figure 6.49d) plus tac (Figure 6.49e), here it's (Figure 6.49f)

(a) (b) (c)



(d) (e) (f)

Figure 6.49. Alberto at the IWB speaking of the sum

Discussion

This new experiment creates in the students a sense that the third line is somehow to be referred to the person who moved (the most), even if it is slightly modified. The students articulate their thoughts by enmeshing the characteristics of the graphs that are now more prominent, namely the resemblance of the third graph with the pink line, and the result of the sum of the initial points in the initial point of the dark blue graph. Mathematically speaking, this experiment realises the sum of any function whatsoever with a constant function as a vertical translation of the given function. Giuliana's comment somehow shades the idea that this is generalisable, and Giulia also doubts the possibilities that lines can only 'obey' to her line.

Alberto, instead, returns again to the idea of the sum, pointing attention to specific positions. He first focuses on the 6 second-instant, where both the pink line and the dark blue line have a minimum; then, he moves to the vertical axis for comparing the intercepts of the graphs. Using both hands, he piles the distances up to get to height of the dark blue line by adding the other two heights. His movements are quick, and he repeatedly points to the same positions or gestures the distance with sudden up-and-down movements. He centres his gaze and body on a single point for each of the three graphs, reducing again the complexity of the whole experiment to a single instant. This aspect brings forth the graph as a collection of still frames. The frames of the three graphs that are in a relationship are vertically aligned, but any point can be chosen, even if Alberto insistently returns to the intercepts, which have catalysed the attention of the class in the previous discussion. Moreover, we see that the third line is threated as homogenous to the other two, still capturing a distance somehow, even though its nature is still indeterminate.

The last concern by Alberto does not convince the other students. The researcher still tries to value his promising idea by asking the students to think of an experiment that could validate or exclude his thought. Thus, a new experiment takes place: after some minutes of planning, Arianna and Giulia alternate back and forth movements going in opposite directions with respect to each other, trying to keep the same displacement as a reference (two tiles on the floor). The graphs are shown in Figure 6.51a. With these new graphs, Alberto again sums the distances: this time, he keeps his index fingers fixed to transfer the distance between the horizontal axis and the blue line above the pink line, so that the upper finger goes to touch the dark blue line (Figure 6.50).

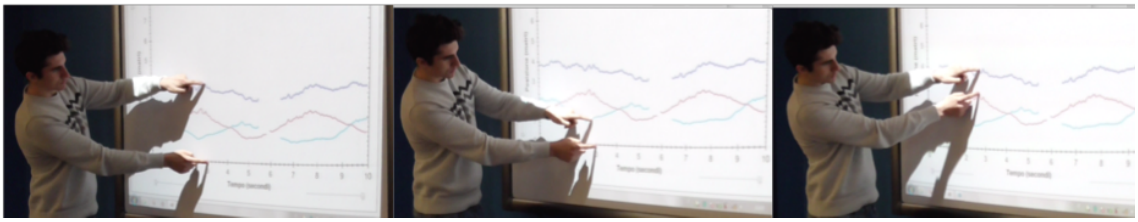


Figure 6.50. Alberto's gestures to sum distances at the IWB

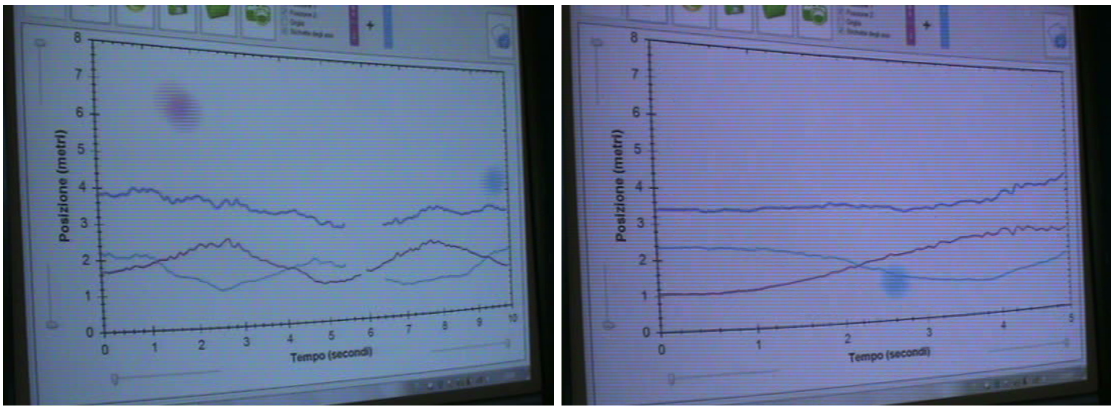
The other students are not completely satisfied yet, and a new experiment is done (Figure 6.51b). Arianna focuses on the intercept and verify that the sum of the distances there is the one she expects (in a way similar to that of Alberto), then she simply adds: "If we take a point, it should be verified elsewhere".

R: On the vertical axis, does the reference quantity change?

Arianna: It's always a distance, in my opinion, and…

Alberto: Yes, it's only that… for example, this notch (*points to a notch on the time axis*), the sensor starts, takes, for example… the sensor captures the distance between the first remote (*points to the interaction space*), between the second remote and then it sums them and traces the dark point, so then many points (*points to the initial point of the dark blue line, then gestures towards the following points of the line*)

Some other students intervene, but the researcher has to close the activity of the day, and the discussion stops.



(a) (b)

Figure 6.51. (a) Sum graphs (3) and (b) (4)

Discussion

In the last part of the episode, the new experiments focus on opposite movements by the two students. It is as if the class wanted to try something completely different from the previous experiments: nor both people should stand still, nor one person should move while the other stands still. Instead, both people should move in some fashion. One can argue that, among the manifold of possible movements that the students could perform, that of moving in opposite ways might be promising: whether they coordinate with each other precisely, they should obtain lines whose sum is constant, since the increase and the decrease of distance compensate for each other. Nevertheless, the emergence of the new configuration does not illuminate any new, particular issue about the graphs for the class, exception made for Alberto, who again performs the summing movement at the IWB. This time his movements are more focussed, he chooses again a minimum and keeps the arms rigid in the physical effort of preserving the distance that he wants to translate vertically on the Cartesian plane. Again, he has little fortune in convincing the classmates, but Arianna uses a similar argument when trying to explain that the sum functions for the initial instant of the last experiment. The episode then closes with Alberto's tentative of extending his argument to all the points of the line. He now speaks of the positions, without reference to a particular time instant, and, at the same time, proposes that that which happens for one extends to all the points.

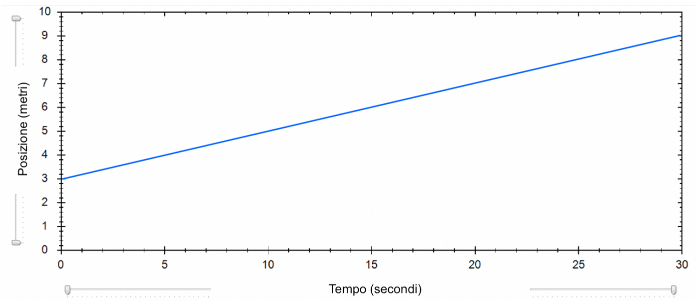
We might say that, at this point of the intervention, the nature of the dark blue line is not completely determined. In this episode, I traced a path along the multiplicity of potential meanings arisen from the experiments with sum graphs. I touched on how different prominent aspects of the graphs, according to the different experiments, bring forth new possibilities for the third line. The experiments are generative in that they create new modes of navigating meanings and move the students to deal with something which is very familiar to them (the sum) in an unfamiliar appearance (the sum graph). In this particular episode, the relationships between the graphs are once more crucial in the way that they direct the provisional meanings that are assigned to the third graph. For example, when the average is the assigned meaning, the being in between of the third line with respect to the other lines guides the disruption of this particular possibility.

By showing the diverse ways that the sum is partially or misleadingly envisioned by the students, even though generative, I also mentioned the insistence with which one student, Alberto, tries to capture the summing of distances. His movements become more focussed as he continues to capture with his own body the holding of relationships and, at the end, his movements reverberate through other students, like Arianna, who coordinate herself with the graphs along the same bodily lines.

6.7.2 Again, on the sum

In the next day, the students start with experiments on the sum modality. Then, they are given a written task to be faced in groups and one individual task (Scheda 6). The individual worksheet was later discussed, at the beginning of the following day. The first question asked the students to draw the sum graph of two given straight lines. The second question was:

2. Now imagine having this straight line as a goal for the sum:



Suppose you can move the controllers only at a constant speed. With which movements can you create this sum graph?

a. Describe the movements and their analogies and differences.

b. Draw the corresponding graphs.

Explain your answer.

In this episode, the discussion is entangled with the presentations of some students' solutions to the given task. The aim is that of bringing forth the powerful ideas that (can) emerge, and the ways of discovering new solutions by means of tinkering with the graphical representations. In particular, I focus on some of the graphs that were produced by the students: these are shown in Figure 6.52 (a to f). We observe two main approaches. On the left column, the three Cartesian planes all contain the operands' graphs as (1) a horizontal straight line and (2) a line parallel to the given line. This is one of the possibilities explored in movement by the students in the collective discussion. Here the graphs cross each other (c) or not (a, e). In graph (e) one of the lines is superimposed to the time axis, while the other one matches the given line. If we look at the second column, instead, lines that are slanted as the given graph can be seen, but not all of them can be the solution for the task: (b) cannot. We first consider the other two configurations. Mostly likely, to draw these lines the students fixed the initial and final distances for the two graphs and drew the graphs by joining these points, which allows for creating the straight lines. For example, we can notice that they indicate with notches or letters the final or starting positions, respectively. It is interesting also to see that these points are sufficient to draw a suitable line, since given two any points that satisfy the relationship, the line is uniquely determined. We can also observe that in both the graphs, one could have also chosen to join the starting (final) point of one line with the final (starting) point of the other line, so that the two lines would have crossed each other; this also constitutes a suitable solution.

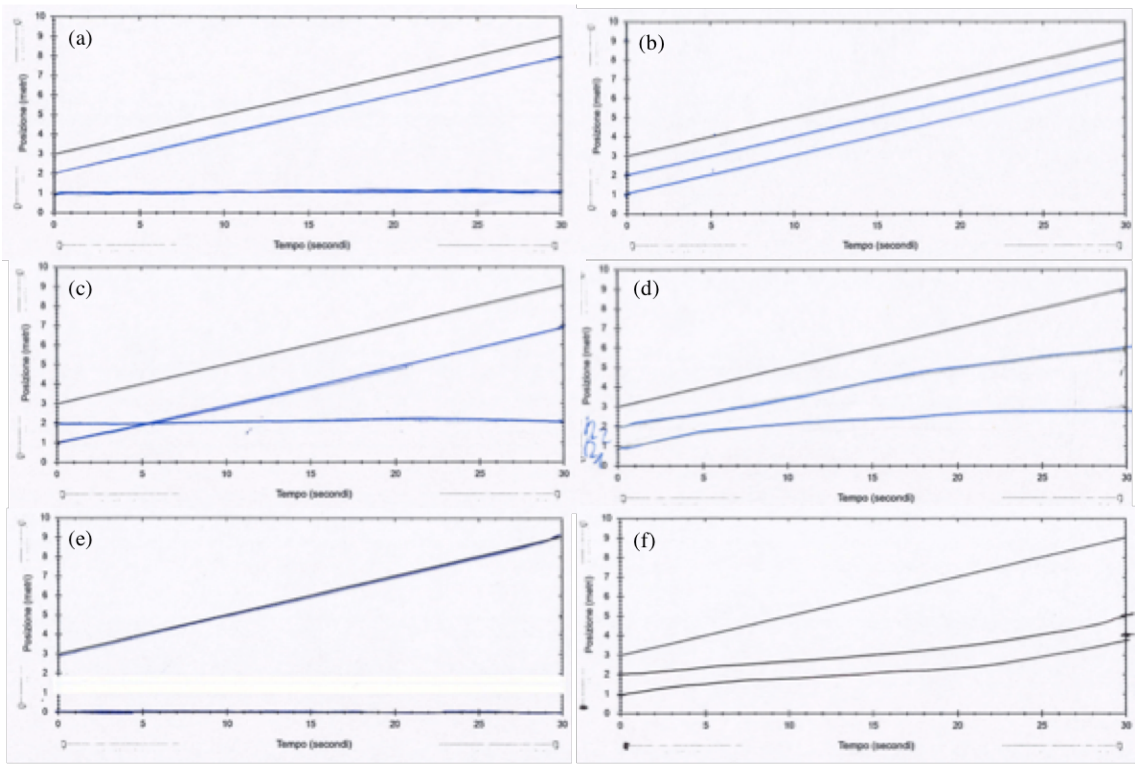


Figure 6.52. (a-f) Some of the configurations produced for the second question of Scheda 6

It is probably an aesthetically-driven choice that of having both lines increasing, as the given one. The non-valid solution is still a very interesting one. In fact, it captures this last issue even more powerfully. I think this is worth mentioning here since it was, in the design phase, source of considerable discussion between the senior researcher and me.

There exist a pair of lines that are parallel to each other and can be summed to obtain the given line, but they are not parallel to it. This is algebraically evident as two parallel lines have the same slope: if we sum the two equations to write the sum equation, we will get a line with double slope with respect to the two parallel lines. But, if two horizontal (parallel) straight lines can be summed to obtain a line which is (still) horizontal, why cannot be the same even here? In the first case the sum graph does not belong to the same family of the operands, while in the case of horizontal straight lines, it does. In WiiGraph, we might think of a third person moving (as in the thought experiment of Alessandro, in the introductory chapter), constrained to the other two (who move in the same way) to represent their sum. Since she will have to double their displacement over time, she will be constrained to moving at a double speed. Therefore, the sum graph will be increasing considerably faster than the other two, to preserve the relationship over time.

During the collective discussion that followed the test, most of these issues were raised and discussed by the students with the researcher, enlarging the possible answer to a wider spectrum, discussing similarities and differences between the approaches, noticing how one solution can become another one via transformations (like the different configurations in the first column of Figure 6.52) Whether we could think of "negative distances" by shifting the reference point, for example, we might also explore new potential configurations (and choreographies), which are not bounded to the limited portion of the Cartesian plane in between the time axis and the given line.

For me, all of this points out the power of having the possibility to encounter mathematical concepts in different manners and with different representations. Also, the interplay of algebraic, diagrammatic and kinaesthetic powerfully emerges as one aspect completes and expands the others.

6.8 *Versus* and collaborative tasks

In this section, I focus on collaborative tasks with *Versus*. As we mentioned in Chapter 4, this modality allows for creating graphical shapes that arise from the composition of two movements (see also Appendix A). Producing a specific plane figure therefore requires coordination between the students' movements. My attempt here will precisely be to examine the ways in which the students encountered this particular type of graph in the context of collaborative mathematical tasks and how movement taps into the event of its creation. Tasks of this kind are also discussed in Noble et al. (2006) and Nemirovsky et al. (2013), where the mathematical instruments at work are different.

6.8.1 Coordinated sympathetic movements

Versus plots an ordered pair of the positions of the two controllers over time, leaving time implicit. Differently from the other modality we described until now, using *Versus* produces a single graph, even though it requires the presence of both the remotes. Indeed, if two users are involved with the remotes, the resulting graph depends on both their movements: vertical displacement in the graph corresponds to one user's movement, horizontal displacement to the other user's movement. A session in *Versus* has no limited duration but can be restarted or toggled to freeze processing when necessary. The most interesting challenges with *Versus* graphs demand the production of known plane figures, like rectangles, rhombuses and circles. These tasks have a collaborative nature, since, when two users hold the controllers, the task of creating a specific figure implies that they have to coordinate their movements over time in a joint effort. In a recent work that emerged out of my previous research study (de Freitas, Ferrara, & Ferrari, 2018), I already explored a collaborative task in *Versus* that asks two (upper secondary school) girls to move the remotes in order to make a circle. In the case study analysed in that work, the video transcript reveals the delicate way in which the two students negotiate a plan of action and different scenarios where their two movements are to be assembled. One of the girls embodies the rocking and rhythmic motion of a pendulum clock, her two hands on the remote swinging to and fro, evoking the continuous and rhythmic movement implicated by the circle as a concept. The two girls struggle with finding ways of negotiating their agency and power. Each of them has to move in a particular manner to allow for the creation of the circle. In the same time, none of them can be completely independent from the other, nor they should do 'the same thing' all the way through. Looking at the micro perceptual nuances of their movements, which should be coordinated but diverse, we theorized the way in which sympathy emerges as the bond that sustains the activity and comes to constitute a way of delving into the nature of collaborative coordinated movements implicated in some mathematical concepts. In the case of the circle, it is not a matter of identification between movements, but more a question of productive tensions implicated by the changing speeds and the sustained but out-of-phase movement (the same movement has to be pursued by both the students, but with some delay). Their timed accelerated movements *are* the gradients that are imperceptible in the graph. The assemblage of graph-concept-student is achieved through these gradients. Their speeds must be different but coordinated for the combined effect. Each hand movement has its own rhythmic pattern, and each hand must move at a different speed, and indeed at related rates of changed speed, in order to achieve the effect. Thus, the two bodies are moving together but apart, and the coupling of these movements forms a third movement that belongs to neither of the original bodies. The various motions inherent in the concept of circle are experienced in the affective bonds that the girls form. It is important to observe that this does not mean that a particular affect is associated with the circle, but rather that a particular experience of affectivity (dynamic coordinated movements at various scales) is associated with the circle.

The productive intensity of the task comes from the various contrasts or tensions that are entailed—there are two girls, each with their own life history; two orthogonal directions to be performed; two very different movements to produce the one graph. Sympathy is the coming together of these contrasts, not so one obliterates the other, but instead as an onto-creative act in which *commotion* becomes coordinated and brings forth new joint learning. This is a task that demands all three components of a sympathetic relation: (1) there is a circulation of feeling as minute facial expressions and changes in bodily posture occur, the two girls leaning in and out, attending to the micro-scale corporeal signals that circulate beneath consciousness; (2) there is a common sense or shared sensibility in the shared obligation to follow each other and work with a shared objective; (3) there is the compassion for the other, and the care of ensuring that others (with different objectives) are coming along, moderating the tensions that sustain any learning assemblage. Sympathetic coordination is the seed of learning because it affords new action-paths across the event and furnishes opportunities for collaborative inventive practices. Concepts emerge and settle in such an environment as a function of sympathy, without a master who legislates the nature of the sense-making (de Freitas et al., 2018).

In the next section, I will also add to this argument a discussion of a first-person experiment in which I am alone trying to make a circle with WiiGraph, in order to tackle the question of the implication of the concept of circle from another perspective. In the following subsection, instead, I offer a brief episode in which the upper secondary school students explore *Versus* graphs for the very first time in the context of a collective discussion. The discussion will bring forth the peculiar contribution played by the concept of square (or rectangle), which the students were asked to create. In the case of this different collaborative task, the affective modulation of mathematical experiences is at the core of my interest, as sympathetic bonds emerge with new nuances, and help us enrich the ways in which the concept does matter.

6.8.2 First explorations with *Versus*

Eleonora and Alberto are the students who volunteered for using the software in a new modality towards the end of the last day of the intervention. Without saying anything, the researcher makes the first session start and a dark blue line appears on the screen. For the first seconds only Alberto moves towards the sensor, while Eleonora stands still in her initial position (Figure 6.53:1). In doing so, he says: "I am the violet" (Figure 6.53, left). Right after, the students both move freely, Alberto walking from side to side of the interaction space and back and forth, and Eleonora simply back and forth (Figure 6.53:2). Alina intervenes claiming: "Practically, there isn't time". A third moment follows, in which Alberto stops and asks Eleonora: "Try to move, you". A vertical trait of line (Figure 6.53:3) is finally produced.

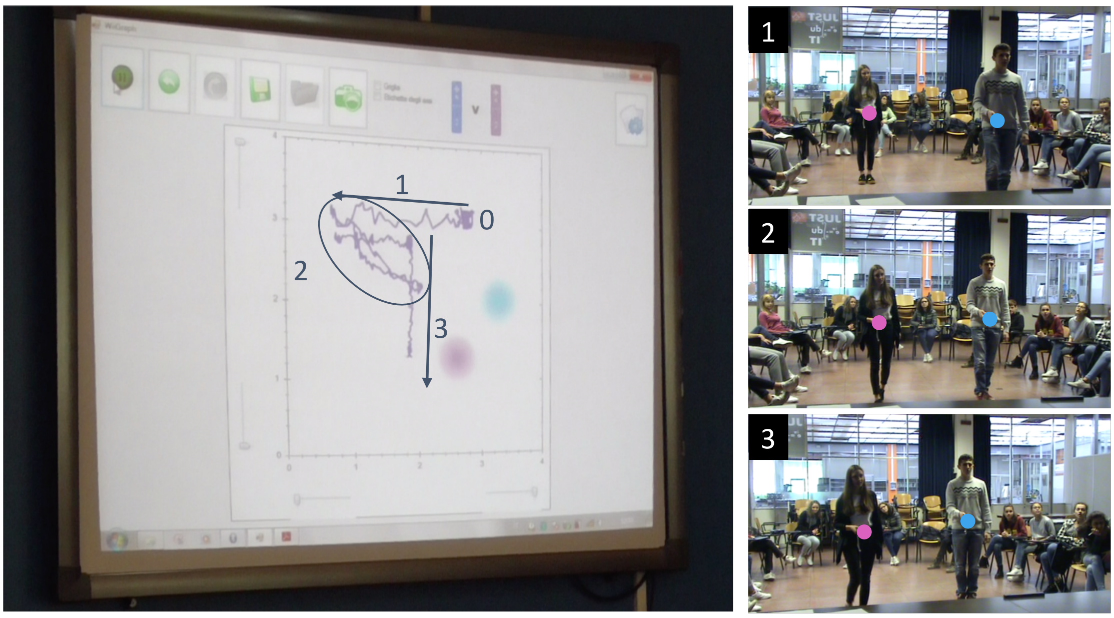


Figure 6.53. First explorations with Versus

Annalia, who intently gazes to the screen while Eleonora moves, suddenly utters with high pitch: “Oh! I understood! One of them moves this way (*repeteadly moves her hand vertically, up and down*), the other this way (*repeteadly moves the same hand horizontally, from left to right*)”. Alina immediately reformulates slightly this thought: “One is the ordinate, the other the abscissa”. The students then move again, increasing their pace and producing other traits of the line on the screen.

It is at this point that the researcher primes a new situation:

(Alb = Alberto, Ali = Alina, Ann = Annalia, Ari = Arianna, Ele = Eleonora, R = researcher, Ss = students)

1. R: Can we make a rectangle?
2. Alb: Ahh, I understand! Practically (*points the remote to the IWB*)… hm, wait
3. Ali: You’re the abscissa (*gestures a horizontal trait*; Figure 6.54a)
4. Ele: You move like this (*almost inaudible*; Figure 6.54a)
5. Alb: I move like this (see Figure 6.54a), you move from below (see Figure 6.54b)
6. Ele: However, parall… (*repeatedly and widely gestures in front of her a displacement up to down*; Figure 6.54b)
7. Alb: They’re the distances, aren’t they? Wait, if we go behind (*turns and moves farther from the sensor, Eleonora follows him*). So… (*gazes at Eleonora*) they’re the distances from, from the sensor
8. R: You’ve to stay closer to each other, otherwise you don’t help each other, at all (*Alberto and Eleonora shift horizontally closer to each other*)
9. Alb: Wait (*rapidly gazes at Eleonora again*) let’s make a try
10. R: Which kind of try?
11. Alb: If I move forward (*moves towards the sensor*), and then I stop (*stops*). You move!

Eleonora slowly move forward and gets close to Alberto.

12. Alb: Ok, always the same line. And then if now I move backwards (*moves backwards to get to the initial position*)
13. Ss: And this isn’t a rectangle! But it’s a square (*rising up, with high pitch*)
14. Ali: You moved in the same space
15. Alb: Go, go backwards

Eleonora slowly move backwards and gets close to Alberto again.

16. Ss: Woow (*a square is now entirely visible on the screen*; Figure 6.55a)
17. Ele: Let’s do a rectangle, now?
18. Ss: Well, now you have to move differently, though
19. R: Why, then?

20. Ari: Maybe to go faster, maybe

21. Ann: No, they have to stop at… (*to Arianna*)

22. R: We saw, we see the rectangle, the square, why does it work like this?

23. Ann: Because they have different… things (*gestures a horizontal line, probably meaning the sizes of the rectangle, speaking to Arianna*)

After some confusion about which of the students is respectively contributing to horizontal and vertical displacement of the line, Edoardo suggests that to make a rectangle one of them has "to move forward more than the other". A new experiment therefore starts:

30. Alb: Go down (*to Eleonora*)

Eleonora slowly move forward, then stops.

31. Alb: Ok, now I leave (*starts moving forward*)

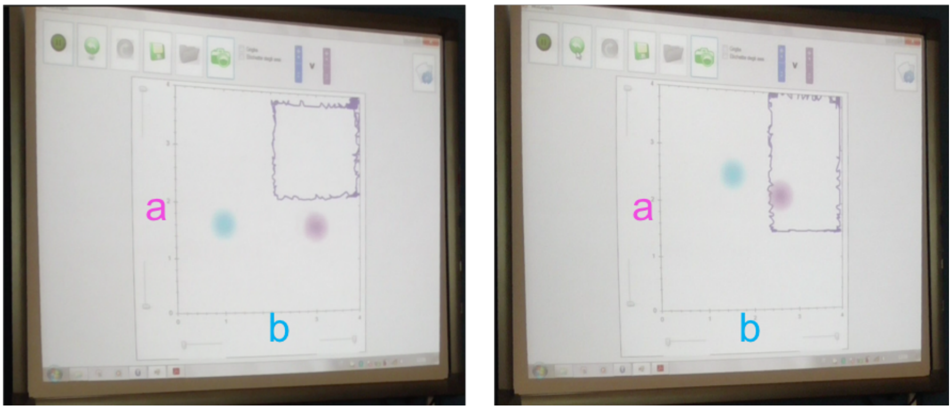
32. Ss: Stop, stop, stop! (*Alberto suddenly stops*)

Eleonora moves backwards again, then stops. Almost immediately, Alberto moves again backwards and reaches Eleonora at the same distance from the sensor. A rectangle is shown on the screen (Figure 6.55b).

Figure 6.54. (a) From left to right: Eleonora, Alberto and Alina gesturing horizontal displacement more or less simultaneously [3, 4, 5]; (b) vertical displacement gestured [5, 6]

Discussion

The students are investigating meanings for the new modality of WiiGraph, which now returns on the screen a unique graph to which two users contribute with their movements. The first free explorations with the controllers engage the students in rich insights into the nature of the new kind of line. At the very beginning, only Alberto moves, while Eleonora stands still. Alberto thus thinks that he sees "his" line (as he utters: "I am the violet"). As soon as the students both move, the line begins producing a doodle, very different from the initial part of the line, which was jagged but almost horizontal. Alberto stops and asks Eleonora to move while he remains steady. A vertical piece is now created by her moving forward. The separation of movements in this third moment immediately prompts the recognition of the different contributes of the two students to the creation of the graph. The novelty is repeated and actualised through gestures and words by different students, who all decompose the movements in the two directions: vertical (ordinate, Eleonora) and horizontal (abscissa, Alberto). The students cannot see the labels of the axes on the screen (they were hidden during the activity and have been added just for the sake of clarity in Figures 6.55a and 6.55b). However, this first experience is sufficient to make emerge diverse ways in which the two students partake in the singular unfolding of the line. In the first exploration, the students are partaking in the event with their own individualities. Alberto moves almost immediately, someone reacts by saying "What's that line?", and Eleonora initially, timidly, stands still. In the second segment, Eleonora also contributes by moving forward, while Alberto seems to wonder around, convinced that the line speaks directly to his lonely movement. The messy shape that results presumably prompts Alberto to stop and ask Eleonora to move. Certainly, the partition of the event of movement in these three moments is only done *a posteriori*, after the experience, which instead flows without clear cuts. This separation, though, helps us better grasp the alternation of movements and contributions, as well as the implicit and delicate distribution of agencies and forces in the situation. We can see that the new situation offers a consistent unbalance with respect to the previous experiments: now the students are two with only one line on the screen. The episode sheds light on the way in which the separation of movements isolates and illuminates the specificity of each student's contribution to the creation of the specific shape and, at the same time, the way in which their entanglement gets to be assembled in the activity.



(a) (b)

Figure 6.55. (a) The square and (b) the rectangle produced by Eleonora and Alberto

The question of the researcher “Can we make a rectangle?” primes a new situation, in which this separation is necessary to assemble the required figure. The students move as sketched in the diagram of Figure 6.56, where the remotes indicate the position of the corresponding student at the end of each part (segment of movement, side of the figure), while the arrows show which remote (student) was moving (its direction and travelled distance) while the other was standing still. Each of these phases allows for the creation of one side of the figure. The sequence of movements is dictated by Alberto’s instructions, and is pursued until the end, resisting the disappointment of most of the other students, who rise up when they see two sides of the expected-rectangle to have almost the same size. Then, the new experiment starts with Eleonora moving forward. Alberto moves as she stops but is blocked by the insistent “stop” of the classmates after a few steps. The rectangle is therefore achieved through a consistent change in terms of power, as one of the students has necessarily to limit his displacement to create uneven sizes.

The separation of movements characterises the square and the rectangle. The four sides of the figure demand interruptions and quick departures. Still, both users are important to create the figure. Differently from the circle, they do not move at the same time, but have to alternate their role in the constitution of the figure. Each of them takes the responsibility to two pieces of the line, which originate from their own movements. At the same time, the absence of movement is also crucial for the creation of the expected figure. Every student partakes with different speeds in the creation of the line. Alberto moves rapidly, Eleonora moves slowly, as if she was handling the line directly in her hand. This way of coordinating movements changes in following activities, when the students are required to move in the same way as if they were producing a rectangle in *Versus*, but with the software showing their position versus time graphs in *Line* (see Figure 6.57). In these moments, they also modulate their speed. No matter the difference in the travelled distance, they slow down and focus more on the overall quality of movement rather than on actual displacement. This is a crucial aspect since it is not that the rectangle (or square) asks for a separation of individualities or intensities, rather the sympathetic coordination is achieved only through an ‘on-off’ relationship between the movements. The concept of sympathy here helps me show a potential way in which this separation might be smoothed: even though in principle the two speeds of movement can be very diverse from each other, as well as the movements happen in distinct phases, the distribution of agency in the activity is sustained by the modulation (not by the identification) of speed and intensities.

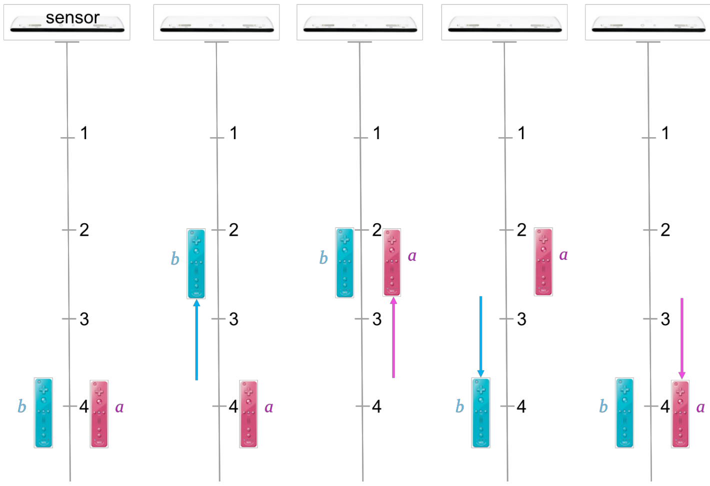


Figure 6.56. A diagram for Alberto (blue) and Eleonora's (pink) making of the square

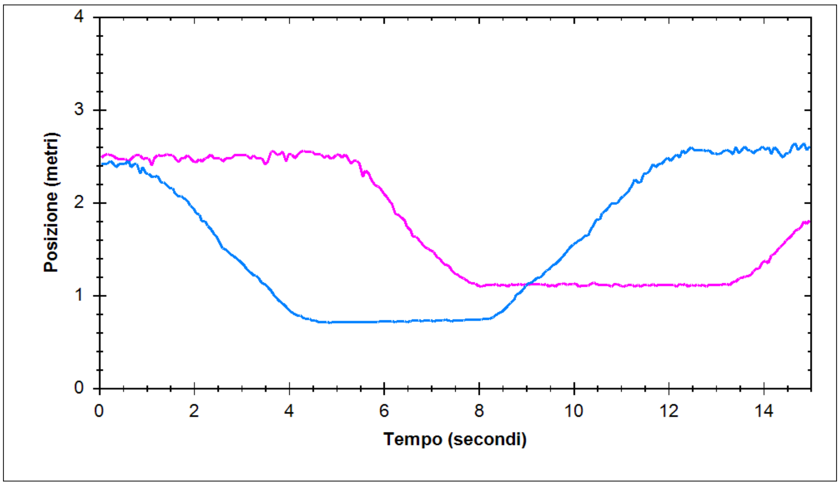


Figure 6.57. Positions versus time graphs for Alberto and Eleonora's movements

6.9 Moving as a circle

In Chapters 1 and 2 we discussed the concept of circle according to different perspectives, namely in the context of inclusive materialism and, more generally, in mathematics education in relation to tool use. In doing so, we highlighted some onto-epistemological issues emerging from such visions. In this final section, we focus again on the understanding of *how it is like* to use WiiGraph, from a different point of view. With this aim in mind, I propose to engage with a first-person experiment in which I hold both the remotes to create a circle with *Versus*. A discussion on the peculiarity of drawing a circle in this way would ‘come full circle’ with respect to previous theoretical positionings, in the context of interest of this study. As a complementary aim, I will use the idea of thinking in movement as developed by Sheets-Johnstone (2011), when describing the dancer in movement from an insider perspective. I will, in turn, draw on my own experience of planning how to make a circle and the subsequent attempts to create it using the remotes and the software. For reasons that should be obvious to the reader at this point, the experience comprises bodily movements in a fundamental way and conjures them with the way in which the technology captures movements through visual outputs. In this introspective analysis of my personal experience, I centre first on the planning of movements and then on the actual creation of a circle with WiiGraph. The two phases might not be so distinct in the usual way of handling the software, however I separate them here for the sake of clarity and in light of a distinctive role that the episode has for the purposes of the chapter.

Planning

I hold the two remotes and I stand in front of the sensor bar, facing the computer screen where the window of the software is open and ready to use *Versus*. I decide to try to make a circle and I start thinking about how to do that. This is more or less a path through my own thinking of a plan. As a mathematician, and being aware of the way the software functions, I know that the circle would emerge from the composition of two parametrical functions that depend on my movements. In particular, since I hold one remote in each hand, my hands would move as if I was producing sine and cosine curves on a position-time Cartesian plane. Then, before I start moving, I decide to take as references for my movement three different positions in space: a central position and two extreme positions. Or better, I focus on the line that I imagine to be projected from the sensor to my body and I approximately fix three different but ordinated positions on that line, one close to the sensor, the second far from it and the third (which is the earliest position I thought of) halfway. The central position will serve me as a reference for the horizontal axis in the Cartesian plane, while the two extreme positions would establish and constitute the limits for the remotes’ movement. In my planning of action, this division of space would allow to adjust the relative positions of hands during my movements (this will become clearer in a moment). Moreover, it will help me modulate speed: for each hand, speed will be at his maximum as I get closer to the central position, while it will be at his minimum as I get closer to one extreme position. Therefore, starting from the extreme position that is close to the sensor and going far from the sensor, I will be accelerating towards the central position and, once passed it, I will begin decelerating towards the farthest extreme position. Both the remotes should follow the same sequence of movements, with the same speed in the same position, but the coordination required would not imply that the remotes should move in an identical manner (i.e., always be in the same position at the same time). In fact, I should take into account that the movements (as the curves) need to be shifted with respect to each other, and that means that (1) I should start with the remotes in different positions and (2) one remote must always be "chasing" the other, passing through the same positions with some delay. Looking at the two sinusoidal functions that I have sketched on a sheet of paper (Figure 6.58), I then realise that there will be some instants in which the remotes will be in the same position at the same time (I am looking at the intersections of the two curves on my diagram). In the same breath, the two remotes will meet while my arms are leading their movements towards different directions, swapping their positions.

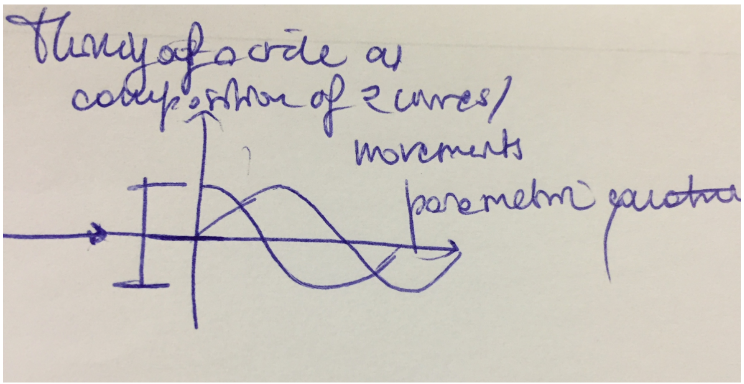


Figure 6.58. Sketched sinusoidal functions, made by the author while planning movements

Creating a circle

At this point I am quite satisfied with my planning of the movements I want to put forward in my experiment. I place my left hand close to sensor, while the right hand is in the central position. I hold the remotes in hand, and I begin moving. In the next paragraph I will try to describe the actual movement of making a circle with WiiGraph as I experienced it, while holding both the remotes. I describe the experience as I remember it afterwards, paying specific attention to the ways in which this experience brings forth peculiar feelings or perceptions for the mathematical concept of the circle.

Since I have to move the arms in different directions, my whole planning presupposed that I would be standing still in the same position, while my arms are moving on the side of my body, which faces the sensor. Walking back and forth while preserving the reference positions would be useless for my purpose other than unnecessarily complex. As I turn on the session, I move with impetus, the left hand chasing the right one. I move back and forth two or three times inside the limited space that I have imagined and restricted around my body, trying to supervise my arms' movement by looking at the remotes. I am not quite satisfied about the result (a messy shape on the screen) as I soon lose hands' coordination. As I start a new session, I realise that the coordination between my arms is easier achieved as I go faster and I do not try to look at the remotes, but I simply focus on the screen feeling the rotational movement in which I feel caught by my own movement. It is as if constraining the remotes' movements along a straight line was limiting my freedom to modulate speed in the course of the remotes' motion. Still, I struggle with keeping the remotes pointed to the sensor, and I try not to pay attention to the blue and pink dots on the screen, which add dynamic elements to the situation. I feel caught up in a strange kind of circular motion, which I enjoy in its repetition, and which occurs to be rotational more than I aimed to, as I achieve the complex rhythm I had planned. I do not stop after creating one almost circular trace, but I go on moving, working hard to maintain hands' coordination. The line continues to wrap the initial circle, creating a thicker and jagged line, but more or less circular.

My body is involved in the arms' movement as my head and torso oscillates back and forth. The head follows the leading hand/remote and the torso follows the head smoothly and rhythmically. The legs, which I did not care about in my planning, are at hip distance, with knees slightly bent, and one leg slightly in front of the other, probably — I realise afterwards — to facilitate my balance in the asymmetric movement of the upper part of my body. The entire movement alternates smooth phases to abrupt transitions, localized in the extreme positions, where the leading hand has to change direction and my body is split up from a concordant to an odd movement. This element brings forth the asymmetry of movement, which is sustained by the slightly rotational trajectory of my hands, so that when one arm moves towards the sensor it is also higher than when it moves backwards, farther from the sensor.

The movement, which at the beginning I perceived as (and I planned to be) smooth and linear, has acquired new qualitative nuances, which I would characterise with unevenness and circularity. Anyway, while moving, I slowly think of my arms' movements as a unique one, which has peculiar qualities on its own and is distinct from the two individual movements; moreover, this movement is in accordance with the originating line on the screen.

As I move, it becomes apparent that some nuances of my movement now shade others, or namely, that they emerge and are predominant in the entire experience. This also reveals that, even if I had planned the movements with considerable attention and deeply relying on my mathematical knowledge, my planning was partial and insufficient.

Of course, this experiment is a highly situated experience of a relatively expert person who uses WiiGraph to create a circle with two remotes. For this and other reasons, I am not arguing that my own experience is transferable to other people, nor that it is the only and unique way of experiencing the creation of a circular shape with WiiGraph. Instead, what this first-person experiment aims at elucidating is how turning to thinking in movement *enlarges the field* when we consider tool use and the emerging properties of mathematical concepts in these kinds of activities. This perspective also adds nuances and details to what is significant in the experience of moving oneself in concert with a technology, in the sense that it shifts the focus to considering what turns out to be peculiar or meaningful in being part of that particular assemblage.

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[1] All the students have a background on kinematics. In fact, in Italian scientifically-oriented upper secondary schools, the students attend physics lessons since the first year (grade 9).

7

Conclusion

In this dissertation I have investigated the role and importance of movement in mathematical thinking and learning in the context of a longitudinal study. Even though I have not offered any definition for such a wide concept, it has been employed in various ways throughout all the work and, indeed, has been the key element that has guided my entire research process. In this final chapter, I close the dissertation by offering the reader an overview of the matter of interest that was unfolded across the pages; by drawing conclusions in light of the research questions; and by unfolding directions for future research.

7.1 From the roots, to the leaves

I enjoy metaphors, especially those that bring into the picture plants and flowers. In this chapter, rethinking of the ecology of this dissertation, I imaginarily move from the roots to the leaves, recovering the effort of making as coherent as possible the research process, which was everything but linear.

An interest in movement has been exposed since the introductory chapter, where I have discussed an episode from a previous study, which brought forth first insights into the relational nature of graphs in WiiGraph. Moreover, the episode fuelled the appreciation of a strict entanglement between the perceptuo-motor, the imaginative and the symbolic in mathematical thinking. These aspects feed one another and are only separable analytically, after the experience, while together they constitute the richness of the mathematical encounters of the students.

Chapters 2 and 3 have been committed to the unfolding of the concepts of virtuality and movement respectively. These concepts have grounded the theoretical framework of my study together with the inclusive materialist perspective (offered inside Intermezzo: Inclusive Materialism), and the vision of concepts detailed in Chapter 1, which specifically focuses on the concept of function. Another Intermezzo has introduced the idea of movement as inspiring a fruitful line of research in mathematics education, especially in relation to theories of embodiment and the body.

Virtuality was fertile ground for discussing how concepts are dynamic arrangements subjected to formation and deformation in non-deterministic ways, as underscored through the idea of multiplicity (following DeLanda in his reading of Deleuze). Drawing on the work of Châtelet, I have highlighted some features of the virtual, namely its obscurity, allusivity and elusiveness, its mobility.

The concept of movement has been further deepened starting from the work of Sheets-Johnstone on the primacy of movement. Her work helped me engage with movement from a wider perspective, as a complex and enigmatic but central concept to grasp life. Understanding movement is crucial from a developmental perspective but comes to get significant especially in relation to thinking. Accordingly, I have explored the subtle process of "thinking in movement" as that in which movement informs and sustains thinking and vice versa. This vision sheds light on the profoundly dynamic nature of thinking.

In particular, thinking of movement through the concept of virtuality, that is, thinking about mathematics as first and foremost mobile, generates questions about how movement enters the realm of mathematics and also the way that moving, or being bodies in movement, partakes in the doing of mathematics.

I propose the idea of *mathematical thinking in movement* as one that brings forth the attempt of capturing the manifold potentiality of thinking with main attention to the bond of movement and the virtual dimension of mathematical concepts. Movement and virtual sustain each other beyond tangling with each other. Therefore, mathematical thinking in movement captures a duality of grasp within the same expression: mathematical *thinking in movement* (that is, thinking in movement in the doing of mathematics) and *mathematical thinking* in movement (that is, the process of thinking mathematically as emerging from movement).

After the description and wide contextualisation of the study methodology (Chapter 4 and 5), in light of the theoretical commitments briefly summarised above, Chapter 6 has offered the main research questions for the study. In that chapter, I have presented and analysed some episodes, which illuminate the potential contribution of the dissertation in relation to the research questions. The insights that blossom from this discussion are summarised in the following.

7.2 Mathematical thinking in movement

The dissertation addresses the overarching aim of characterising mathematical thinking in the context of graphing motion with WiiGraph. This aim has been pursued along the lines of the three main research questions. In this section, I discuss answers to these questions by recovering the analyses carried out in the previous chapter.

The first question investigates the experiences of the students with WiiGraph:

How is the mathematical experience of students using WiiGraph to explore spatio-temporal relationships playing out across entwinements of perceptuo-motor, symbolic/diagrammatic and imaginative aspects of the activity?

First insights from the pilot study (§6.2) evidenced how, using WiiGraph, the immersion in the Cartesian plane is lived and experienced simultaneously in terms of relative motions (persons/controllers) but also in terms of moving lines that interact with generative relationships with respect to each other. In particular, in the interview of Luca (6.2.2) I have discussed these aspects by drawing attention to how the flow of movement implicates dynamic thinking about pairs of graphs and their relations, being generative of mathematical meanings, beyond its own meaningfulness.

To characterise the mathematical experience of the students, who used WiiGraph in this study, focus has also been driven to their experiences of movement. In analysing the three exploratory experiments (§6.3), devising and using a tailor-made movement notation, I have offered the reader an in-depth description of the experience, concentrating the analysis on the perceptual and material engagement of the students with the technology.

Exploratory experiments are, by definition, the ones with more freedom of movement. Nevertheless, a point that the analyses brought forth is the mutual influence that is established between movers, realised in the tendency to look at the other or follow the other, but also in the search for differentiation, even with abrupt changes.

The use of a notation helps bridge the gap between the video data of the experiments and the written page, where the dynamic of movement has to be reduced to words. It also encourages to didactically value movement: enlarging the field of what is or might be significant can be crucial in the common practices of the mathematics classroom. I have suggested the potential contributes of this notation in revealing to us aspects and nuances concerning the experience of moving with WiiGraph, which the graphical notation cannot fully capture. In fact, my analyses do not aim at generalising the prototypical experiences of exploratory experiments. Rather, they aim at offering possible interpretations and visions that take into account the qualities of these experiences, which are very diverse from one another in their overall nature, and they attempt to do so through a detailed account of multiple variations inside movements and interactions.

I have also provided a wider contextualisation of these experiments in the ecology of the class, showing how the graphs emerge as a collective event, which enlarges and grows through multiple bodies. The collective engagement of the classmates that are partaking in the experiment with laughs, gestures, directions and suggestions, while observing the experiment, extends the individual and pairs' perception to a broad sense of *affective attunement* to the exploratory event. Throughout highlighting passages from the collective discussions, I have interpreted the students' interventions, deepening the ways in which WiiGraph entails perceiving the graphs as a couple, therefore in relational terms. Regarding mathematical thinking in movement, I have investigated how exploring becomes playing with qualitative alterations of the given situation; imagining, in the already given scenarios, potentialities that perturb the past experiences; caring about the queer aspects as well as the mostly unnoticed parts of the graphs. All these aspects become significant (for the teacher and for the researcher) when we turn to thinking in movement as a unique process, recognizing meaning to movement itself.

Focusing attention to specific graphical configurations (§6.4), I have discussed the strategies that the students brought forth when dealing with tasks that ask for imagining and actualising (with the diagrammatic and the bodily, beyond words) relationships between graphs. Particular attention has been directed to the qualitative aspects of motion entailed by those strategies and to speed (further examined in relation to the following question).

The second research question specifically draws on a vision of mathematical concepts as material arrangements, which get to be constituted through practice (see Chapter 1). The question is the following:

Which mathematics does it emerge out of the activity? Or better: How does the mathematics change in the encounter of the students with the software?

In Chapter 1, attention was given specifically to the concept of function and to particular diagrams and mathematical instruments, in order to generate a discussion on how the latter engender the concept in specific ways, for example demanding peculiar movements and creating *felt quality* about a function, which is not emerging in the same way through other representations. The question follows this line of flight, focusing on the particular mathematical instrument used in the study.

In particular, I have delved into the relational nature of graphs as emerging throughout all the episodes. In section §6.6, I have looked to the concept of speed, as an intensive quality of movement that plays a crucial role in the interpretation and understanding of spatio-temporal relationships. I have pointed out the emergence of ambiguity concerning speed and I have demonstrated the many ways in which this was generative (§6.6.3), problematic (§6.6.1) and meaningful (§6.6.2) within the mathematics classroom.

In section §6.7 I have turned attention to sum graphs. The analysis captures the unfolding of meanings for a third graph, which is obtained as the sum of two positions over time and is experienced by means of motion experiments. I have traced a path along a multiplicity of potential meanings that arose from the experiments with the sum graphs. These experiments are generative in that they create new modes of navigating meanings and move the students to deal with a mathematical idea, which is very familiar to them (the sum), under an unfamiliar guise (a sum graph). In this specific episode, the relationships between the graphs direct the provisional meanings that are assigned to the third graph. For example, when the average is the assigned meaning, the in-betweenness of the third line with respect to the other two guides the disruption of this particular possibility.

In line with this discourse, and drawing on first insights from the pilot study, I have also investigated a specific configuration:

Specifically, regarding the event of crossing lines: What kinds of meaning does this event generate in the mathematics classroom for the concept of function?

In our discourse, I have proposed that the event of crossing lines in WiiGraph is at the same time pivotal cognitively and generative ontologically. On the one hand, it does allow for connecting each single conjecture by means of a single experiment and for creating the conditions to interpret the graphs in terms of their mutual relationships, that is, in relational terms. On the other hand, it explodes the virtuality of the objects that populate the Cartesian plane: lines potentially cross each other and points emerge out of their potential crossing. These ideas are expanded with the analyses in §6.6.2 and in §6.5, through examples from all the teaching experiments.

The last research question investigates the collective dimension of the activity, by asking:

How does a collective movement of thinking emerge and get distributed across the learning assemblage?

This question was deepened by attempting to show how the graphs are collective productions; by capturing the unfolding of mathematical thinking as it developed across the classroom during and after motion experiments; and studying collaborative mathematical activities in relation to sympathetic bonds emerging out of movement.

Finally, the last section (§6.9) contains an unusual episode, namely a first-person experience of moving with WiiGraph. I describe and analyse my planning and experience of making a circle using *Versus* and holding two remotes at the same time. Presenting a highly situated experience, I claim that the use of the instrument can be mathematically significant even for the (expert) mathematician. The episode illuminates qualitative nuances and details in the experience of moving in concert with a technology, as a way of enriching the potential meanings that emerge from the specific practice. I purposely focus on the qualitative emerging structure of my own movement, describing how the particular concept of circle is implicated in, and infused with, perceptions, surprises and new discoveries. Regardless of whether the learning assemblage is reduced to me, the technology and the circle, assembling with the concept entails that the process of thinking in movement is perfused with, and sustained by, affective bonds and that the experiment creates new meanings and possibilities through movement.

7.3 Future research and open questions

An additional question, which traverses the main research questions and the whole work, concerns the methodological challenges of dealing with movement. The concept of movement helped navigate the landscape of mathematics through concepts, diagrams and instruments, through the lines of the body and through virtuality. At the same time, great effort has been put to finding ways for capturing movement, despite its elusiveness and omnipresence.

When I think of movement, I think of an inescapable and ever-changing flow; of becoming and change. How can we deal with these? Mathematicians have been finding marvellous and smart ways of crystallising and capturing transformation, change and little variations in mathematics. Are these ways still suitable to understand mathematical thinking? I have pursued the use of a movement notation to initially address this question. In the episodes' analysis my attempt is to grasp, through the notation, the micro-perceptual variations that populate each movement. In doing so, I challenge a representational vision of the body, without making claims about what the movements *mean* for the activity, but rather focussing on *how the movements express qualitatively the dynamic of mathematical thinking*.

I am aware that it is only a partial and tentative answer to a wide and problematic methodological question, but is indeed a fertile ground for future research, especially concerning the search for alternative ways of capturing movement and possible modifications, adjustments, improvements for the outlined notation. The pedagogical aim of a notation has not been fully exploited in this dissertation and represents another issue which might be investigated. I am also interested in the ways in which a movement notation exploited by learners themselves might be used to make sense of the graphical notations. Lastly, concerning this point, the notation has not been exploited at all to investigate gestures, or other bodily activities in particular, and further studies could investigate the feasibility of its potential use in this sense.

From a methodological point of view, some improvements could be done regarding data collection. We have used the software with a relatively old personal computer that, for example, did not allowed for automatically recording the software's window. Sometimes the capture of video data was problematic (e.g., the quality of the video recordings was influenced by poor light conditions, after that shading the classroom was necessary to avoid light interferences with the infrared technology). Additionally, in the analysis phase, I have realised that other viewpoints could have been useful to map the embodied interactions with the software, for example top views of the classroom floor. For practical limitations, this was neither possible nor designed in the case of this study. In future research we could take advantage of new technologies, which attend to new kinds of data collection, even reconfiguring the *sensory boundaries* of the researcher in dealing with data.

Moreover, many of the available data have been scrutinised but not discussed in this dissertation, and further analysis might shed new light on the mathematical activity of the students.

From a theoretical point of view, in drawing on the work of Sheets-Johnstone, I did not tackle the importance she gives to consciousness while discussing self-movement. In proposing a first-person experiment in §6.9, in fact, emphasis is more on highlighting qualitative nuances in the moving-thinking with the technology than on consciousness *per se*.

In light of the metaphorical language with which the chapter begins, all these open issues constitute new blossoms that are still waiting for opening up.

Appendix A

This appendix presents WiiGraph, the software that has been used in this research study for the graphing motion activities. These pages complement the methodological aspects already addressed in Chapter 4 and provide the technical information about how the software functions and can be used. We start with an introduction that keeps trace of the developments of WiiGraph and, briefly, of how they are entangled with the study.

WiiGraph: An introduction

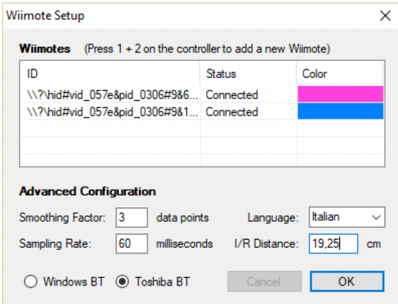
WiiGraph is an interactive software application designed and developed by Ricardo Nemirovsky, Coram Bryant, Michael Meloney and Bodhan Rhodehamel at the Center for Research in Mathematics and Science Education (CRMSE) of San Diego State University (SDSU). WiiGraph is a particular mathematical instrument that allows for the creation of graphical expressions by means of kinaesthetic interactions with some devices. By "mathematical instrument" we mean a material resource, interactively used by means of individual or collective continuous body movements to obtain and transform mathematical expressions (Nemirovsky et al., 2013). WiiGraph in particular belongs to a family of mathematical instruments based on motion detection, of which the MIT-P (Abrahamson & Sánchez-García, 2016) is also an example. These instruments fundamentally involve the body in the production of representations: they work at body-scale involving wide body movements like walking in space or overarm gestures and are responsive to one or more movements occurring simultaneously, whether performed by one or two people at a time.

WiiGraph is the mathematical instrument, which is at the core of this research study, as we have outlined in Chapter 4. Briefly, it permits graphing motion by capturing over time the distances of two controllers from a sensor and creates different representations of this data. Initially, the version 0.9 of the software was privately released (18 October 2012), in the context of the Summer 2012 Professional Development Course "Shadows and Body Motion: Using Technology to Teach Ratio and Proportions" organised by CRMSE and SDSU for teachers coming from around San Diego. The teachers that took part in that course had access to the software, its user guide and the related proposed activities. Later, as mentioned in the Introduction, I carried out a teaching experiment with the supervision and collaboration of Prof. Francesca Ferrara, in which we mainly used WiiGraph with a class of grade 9 students to introduce the concept of function via a graphical approach. After that first experiment, prof. Ferrara and I began collaborating with Prof. Nemirovsky and his research group at CRMSE, by proposing changes and novelties to address some difficulties that we experienced within the mathematics classroom[1]. In the current research, we were thus able to use a revised version of WiiGraph, which is not yet publicly available, which includes the discussed changes (for example, we got an Italian version of the software). The group of the involved people is at present working on a public release for a more reliable version than that tied the Wii system's constraints. In the following section, the functioning of WiiGraph is detailed.

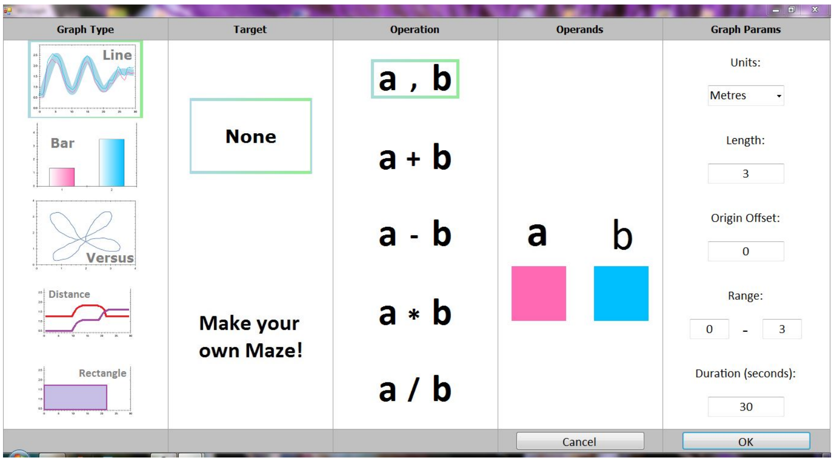
WiiGraph: Devices and options

WiiGraph uses some devices of the Nintendo Wii game console, that is, the sensor bar and the remote controllers (Wii Remotes or "Wiimotes"), connected to a computer that runs the software. The controllers are connected to the computer via Bluetooth technology and communicate with the sensor bar via Infrared technology. Specifically, we made use of WiiGraph on a personal computer running Windows 7, with the USB 2.0 Bluetooth adaptor AZIO (which works with the Toshiba Bluetooth Stack application), a wireless LED sensor bar and two Wii Remotes Plus (one pink and one blue). The combined technologies allow for wireless detection, capture and display of the controllers' position with respect to the sensor bar, with a high degree of freedom in relation to the location of the computer. Nevertheless, it is worth noting that Bluetooth allows for exchanging the data over a short distance (no more than 8 metres, with decreasing accuracy corresponding to increasing distance) and uses radio waves, so it is prone to interference. In addition, the communication between the sensor and the remotes might be influenced by light conditions, as sun waves or projector lights might interfere with the infrared technology of the remotes and the LED signal sent by the sensor bar.

When the remotes are connected to the computer and the software is launched, the window in Figure A.1a opens: it shows the status of each Wiimote and the colour assigned to it by WiiGraph from that moment on, and allows for setting the Language (English or Italian), the infrared distance (that depends on the used sensor bar, which was 19,25 cm long in our case) and the smoothing factor and sampling rate that regulate data capture.



(a)



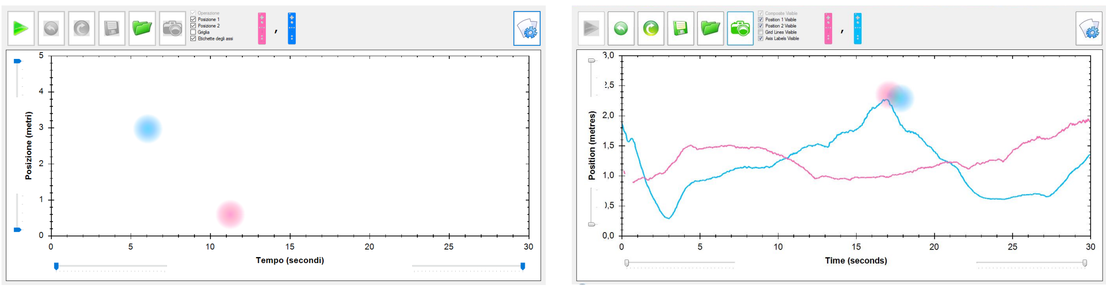
(b)

Figure A.1. (a) Wiimote Setup window; (b) WiiGraph initial window (English version)

After this preliminary window, the initial window of the software appears (see Figure A.1b): on the first column the different possible working settings (Graph Type) are displayed, that is, *Line*, *Bar*, *Versus*, *Distance* and *Rectangle*. The second and third column differ from Graph Type and contain various targets (*None*, *Maze*, *Value*; *a,b*, *a+b*, *a–b*, *a·b*, *a*/*b*; *Perimeter*, *Area*). The fourth column shows to which colours the remotes (Operands) are mapped. The last column permits to modify the graphical parameters (Graph Params) regarding the unit of measure with which distance is to be represented, the range in which the remotes' positions can be detected or displayed and the duration for each experiment with the software. In our study, we always took the sensor bar as origin of the reference system, but it is also possible to select an offset of the origin point. In addition, we used the software inside the mathematics classroom as detailed throughout Chapters 4 and 5, taking advantage of a large interaction space in front of an Interactive White Board, which projected WiiGraph interface on a large screen. In the following subsections, I present the available options of WiiGraph, devoting more attention to *Line* and *Versus* graphs, which were the ones exploited in the study.

Line

Line graphs allow for the creation of position versus time graphs that depend on the distance of each remote from the sensor bar. In the standard modality of *Line*, the displayed graphical window is that shown in Figure A.2a. Labels on the axes (“Position (metres)”; “Time (seconds)”) as well as a Cartesian grid can be displayed or hidden. When two users point the remotes to the sensor bar, two coloured dots are shown on the screen (Figure A.2). In order to collect data for a Wiimote, the corresponding coloured dot must be visible on the screen, but the dots are only an indicator of detection and do not inform the position-time graphs (i.e., students only need to focus on the dot to the extent that they know it is there and their remote is being detected). This also means that the dots are not influential in the creation of *Line* graphs.



(a) (b)

Figure A.2. Line graph’s window (a) at the beginning and (b) at the end of a session

When a session starts (by clicking the play button on the left upper side of the window, or by pushing the “A” button of one or the other controller), two lines originate on the screen, moving from left to right and plotting the position of the users over time. Each line matches the colour previously assigned to the remote. Each can be hidden (or displayed) at the beginning, during or at the end of the session (Figure A.2b).

By selecting an *Operation* on the third column of the initial window, it is also possible to have a third line on the screen: the graph produced by applying the selected operation to the two lines given by the users’ movements. This new graph lies on the Cartesian plane together with the pink and blue graphs but is differently coloured. In the case of Figure

A.3, the selected operation is the sum of the two controllers' positions, which turns out to be a third dark blue line produced on the screen.

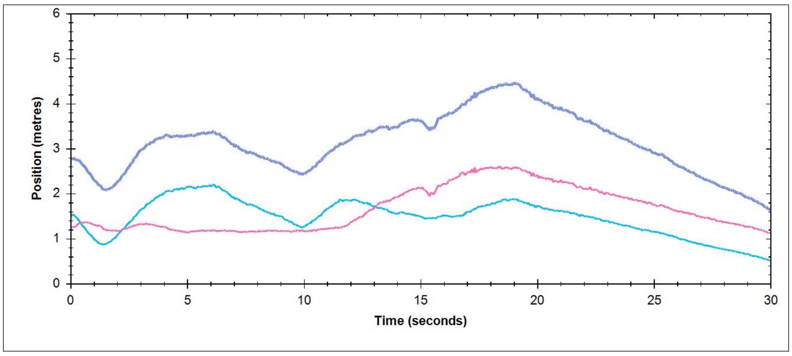
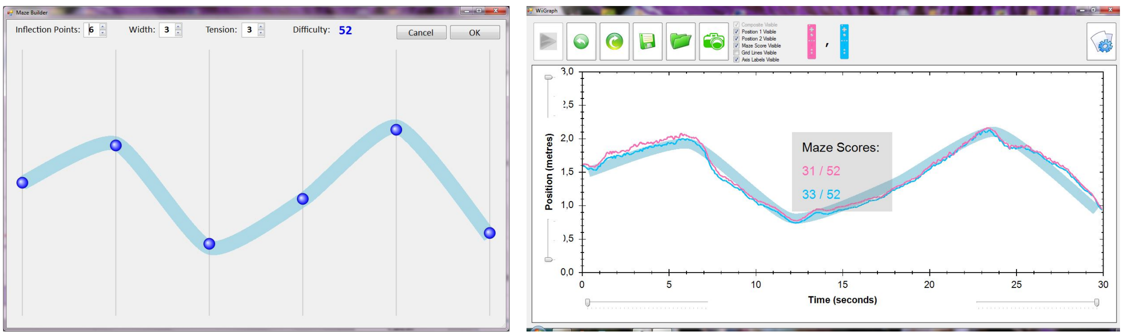


Figure A.3. A sum graph in Line (dark blue line)

When choosing the *Make Your Own Maze!* modality on the second column, it is possible to create a target graph in a Maze Builder window (Figure A.4a), according to specific parameters and characteristics: number of inflections points, greater or lesser thickness of the target line, and tension (sharp-cornered or soft curves around inflection points). Once the target graph is built, it appears on the Cartesian plane and remains visible for the entire session to be traversed by the users. At the end of a session, the software permits to show (or hide) a final score for the participants, which is computed according to the frequency with which each line traverses the target (Figure A.4b).



(a) (b)

Figure A.4. (a) Maze Builder window and (b) a session played with Make Your Own Maze!

Make Your Own Maze! can be used whether with two lines graphs or with one among the available operations. In the latter case, the target is returned with respect to the operation graph.

Versus

Versus graphs allow for the creation of a single graph on a Cartesian plane with isometric axes (Figure A.5). *Versus* plots an ordered pair of the positions of the two controllers over time, leaving time implicit. A session with *Versus* has no limited duration but can be restarted or toggled to freeze processing when needed. If the remotes are held by two users, the resulting graph depends on both users' movements: vertical displacement in the graph corresponds to one user's movement, horizontal displacement to the other user's movement. Even though it is possible to have a target maze, more interesting challenges for a *Versus* graph include the creation of plane figures. For example, in Figure A.5, we see an attempt by a single person to produce a circular figure. When two users are holding the remotes, the task of creating a specific figure is challenging, since they need to coordinate their movements over time.

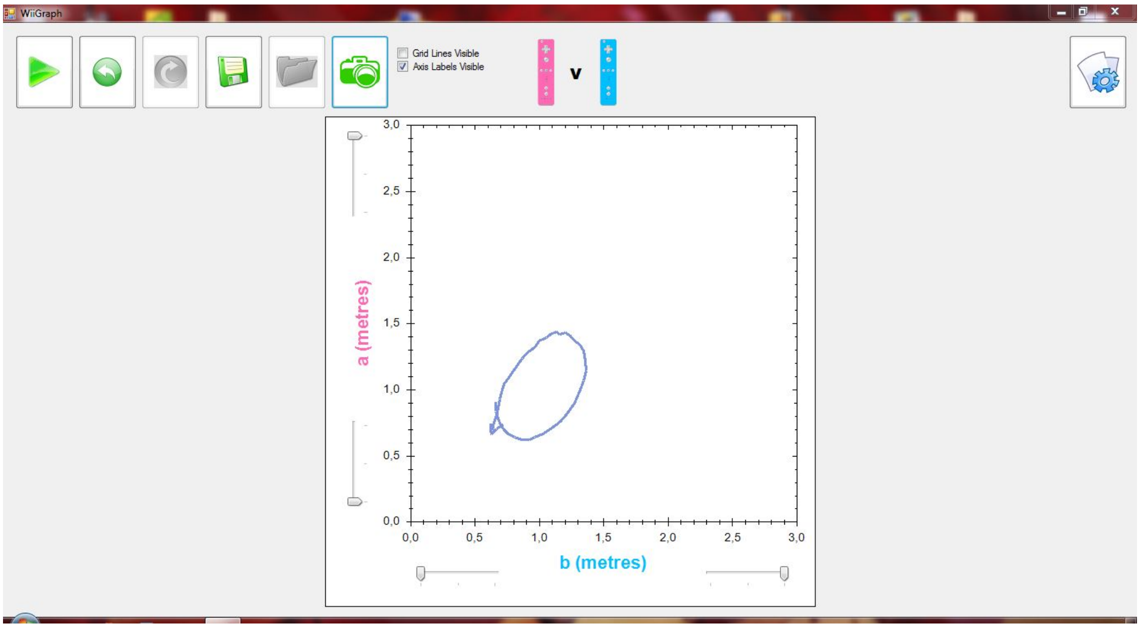


Figure A.5. Versus window with a circular graph produced using the remotes

Bar

Bar graphs plot the distance between each remote and the sensor bar through a vertical bar (column) with varying height. As long as the controllers are directed at the sensor bar, vertical bars will be displayed, coloured accordingly to the associated remote. *Bar* graphs implicate a Value as a target, and the only Operation '/' (ratio). Briefly speaking, the goal in using this option is to work together to achieve the target with the ratio of the remotes' distances from the sensor bar. This ratio is captured by a blue point moving on a horizontal number line depending on the changing positions of the users, while a box indicates the target (in Figure A.6 the target is 2; the figure shows two possible combinations that satisfy the target). Like in the case of *Versus*, there is no limited duration for a session, but it can be stopped or restarted. The Cartesian plane diffusely turns from white into yellow when the blue point is getting closer to the box on the number line, that is, when the ratio is getting closer to the target ratio. Learners can be interestingly challenged to achieve the given ratio and to move while preserving it over time.

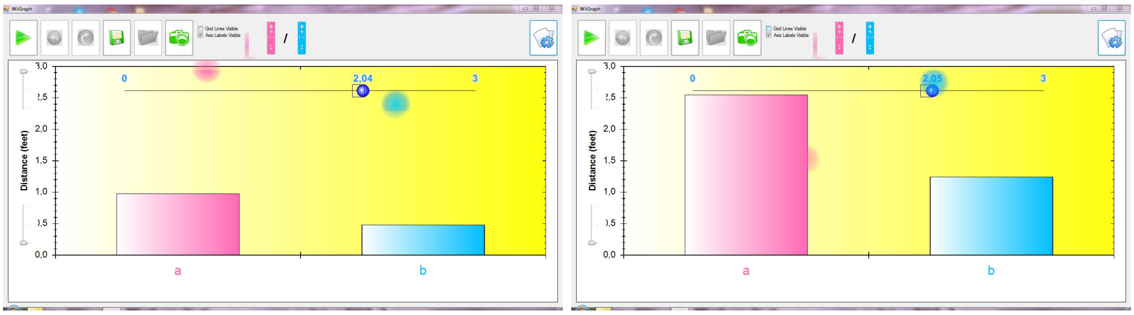


Figure A.6. Bar graphs: two situations that satisfy the target value (2)

Distance

Distance shows the position-time graph according to the movements of a user with a remote, like in the case of *Line* graphs, and adds a graph of the overall displacement over time (Figure A.7). This option can only be used with one remote at a time.

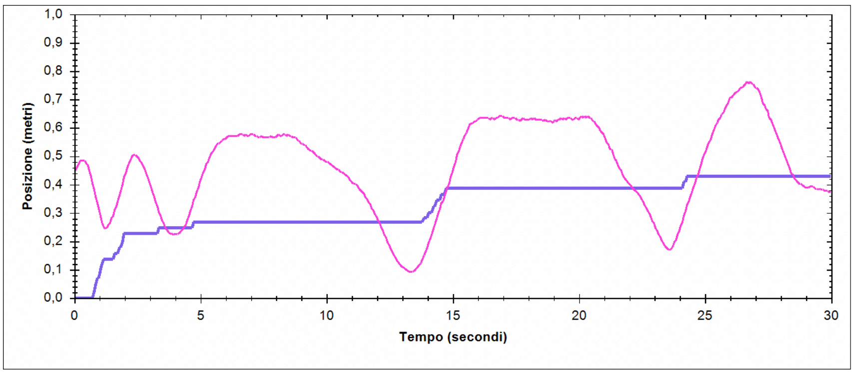
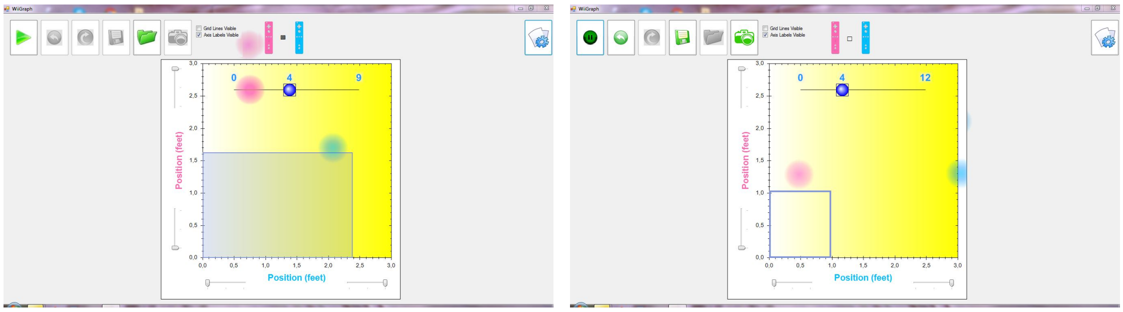


Figure A.7. Distance graph

Rectangle

Using *Rectangle* graphs, it is possible to select a target Value again and to work with the Perimeter or the Area of families of rectangles. When a session starts, the remotes' distances from the sensor are plotted on the Cartesian plane as the two dimensions of a rectangle (on vertical and horizontal axes respectively). A target can be set for the value of the perimeter or the area of the rectangle. The window is similar to that of *Bar* option: the current value for the area/perimeter is captured by a blue point that moves on a horizontal number line following the changing positions of the users, while the target is provided by a box; also, the background gets yellow as the target is reached. Figure A.8 presents two different sessions, working with target (4) area and perimeter respectively, in the moment in which the exact value is achieved.



(a) (b)

Figure A.8. Rectangle graph: (a) Area and (b) Perimeter

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[1] At that time, the use of WiiGraph technology was partly supported by a grant assigned from the National Science Foundation to San Diego State University (Grant DRL-1323587).

Appendix B

This appendix contains the original worksheets and written tasks that have been faced by the classes involved in the research study (questionnaires and final tests included). These tasks are contextualised and discussed inside Chapter 4. Here I present them precisely in the format in which they were given to the students during the interventions (therefore, the reader will find Italian language). In the case in which requests of original worksheets were given in more than one page, I have grouped them in a single page for the sake of space.

Grade 4

Scheda 1

Data ________________________

Nomi: ____

Oggi abbiamo fatto alcuni esperimenti con i telecomandi.

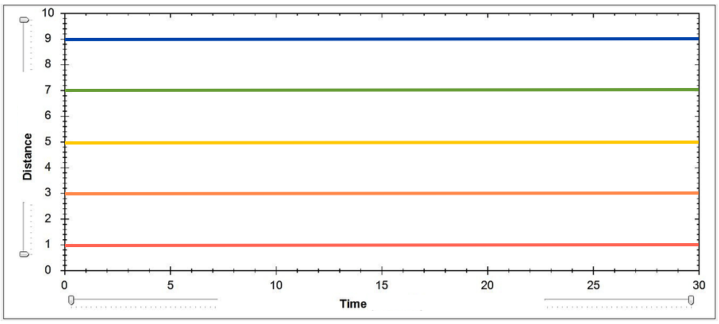
Immaginate di spiegare a un vostro amico che non ha fatto questa esperienza che cosa avete capito, soprattutto di matematico, usando WiiGraph.

Scheda 1

Data ______________________

Nomi: ____

Immaginate di avere 5 telecomandi a disposizione e di vedere questi grafici sulla LIM:



Spiegate come fareste per ottenerli.

Scheda 2

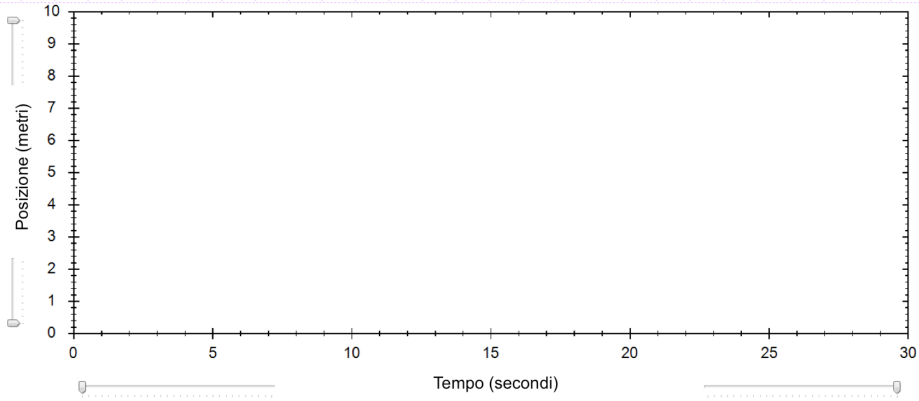
Data ________________________

Nome: ____

La volta scorsa abbiamo visto come possiamo ottenere linee orizzontali parallele usando i telecomandi.

Adesso, Bianca e Andrea vogliono fare un nuovo esperimento: vogliono ottenere due linee **oblique** parallele usando i telecomandi. Che cosa devono fare?

Scegli due linee oblique parallele per l'esperimento di Bianca e Andrea e disegnale. Poi, scrivi tutte le informazioni e indicazioni importanti da dare ai due bambini per ottenere queste linee con WiiGraph.

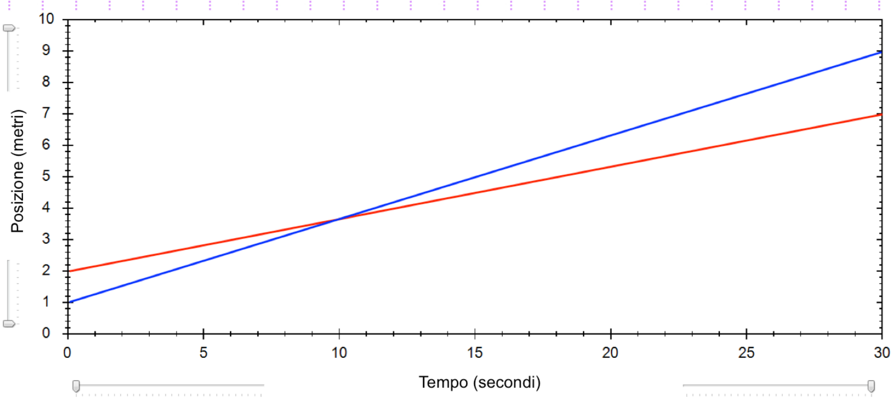


Data ________________________

Nomi: ____

Rob e Bob sono due robottini che fanno degli esperimenti con WiiGraph.

Immagina di entrare nell'aula in cui un esperimento di Rob e Bob è appena finito. Sulla LIM trovi queste due linee:



Racconta come si sono mossi Rob e Bob durante il loro esperimento.

Scheda 4

Grade 7

Initial questionnaire

DATA ___________________ NOME ____________________________

Lavoro individuale

Questionario

Ti chiediamo di rispondere a poche e semplici domande nelle prossime pagine. Usa tutto quello (parole, disegni e altro) che ritieni necessario per essere il più chiaro possibile nelle tue risposte.
Ricorda che in questo caso non ci sono risposte giuste o sbagliate.
Vorremmo imparare a conoscerti fin da ora, con questo questionario.

1. Hai mai sentito parlare di grafici? Quando? Che cos'è per te un grafico?
2. Immagina di osservare due bambini che si sfidano in una gara di corsa: al via, Andrea parte veloce e Bianca più lenta. Poco dopo, Andrea è costretto a fermarsi per allacciarsi una scarpa. Bianca lo supera, e taglia il traguardo per prima.
   Se dovessi spiegare la storia che racconta la loro corsa con un disegno, come faresti?

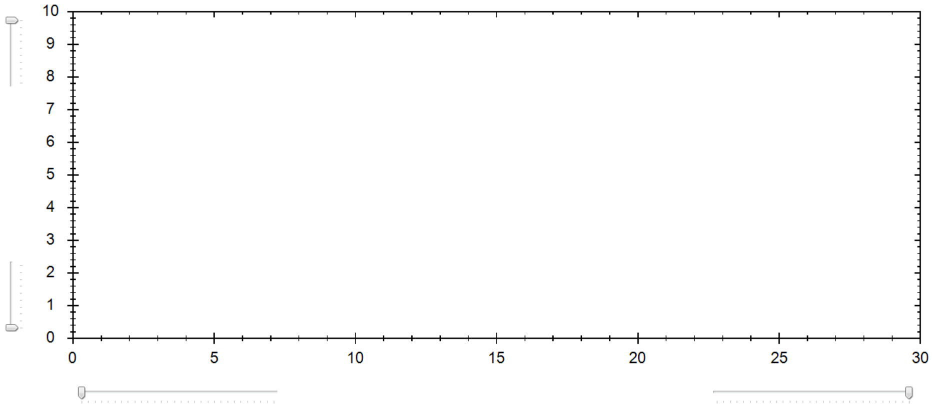
Questionario 1

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 1

1. Disegnate due linee (non identiche tra loro):



Descrivete a parole i due movimenti che, secondo voi, permettono di ottenere queste linee con WiiGraph.

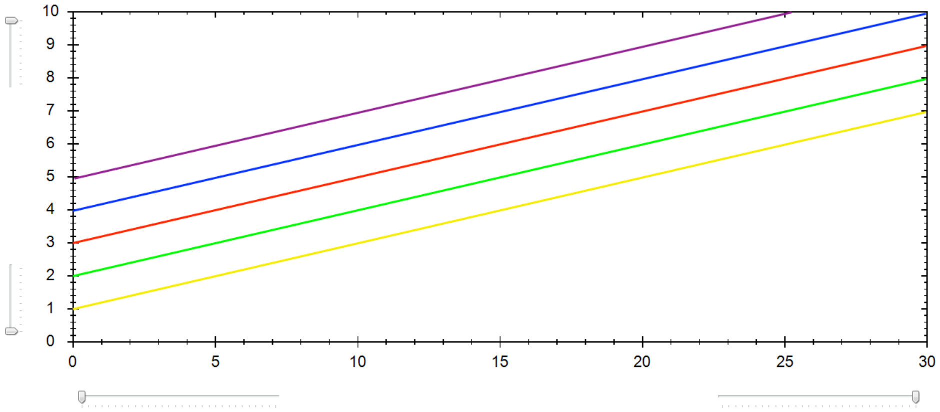
2. Avete a disposizione al massimo due tentativi con WiiGraph per provare a ottenere le linee che avete disegnato. Spiegate come modifichereste la descrizione dei movimenti dopo queste prove.

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 2

Immaginate di poter utilizzare più di due telecomandi e di ottenere cinque linee rette:



1. Spiegate con quali movimenti si possono generare queste linee, facendo attenzione a specificare tutte le informazioni che ritenete importanti.

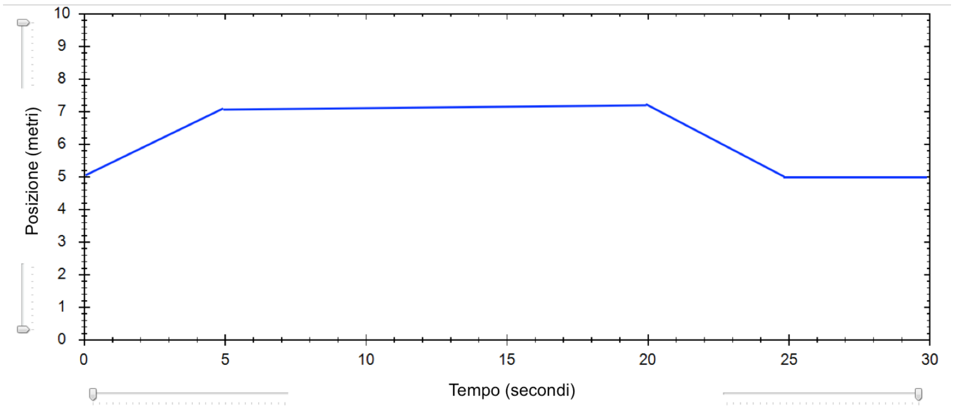
2. Secondo voi, che cosa hanno in comune le cinque linee? Che cosa invece hanno di diverso? E i movimenti?
   Spiegate sempre i vostri ragionamenti.

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 3

1. Rob e Bob sono due robottini che sono stati programmati per muoversi in modo molto preciso. Immaginate che facciano insieme un esperimento con i telecomandi e che WiiGraph produca questa linea in corrispondenza del movimento di Rob:



Anche Bob si è mosso, ma la sua linea è rimasta nascosta!
Sappiamo solo che Bob è partito assieme a Rob, alla stessa distanza dal sensore, ma si è mosso sempre a velocità doppia e nel verso opposto.

- Secondo voi, quale linea mostrerebbe WiiGraph per il movimento di Bob?
- Rob e Bob, una volta partiti, si sono incontrati ancora?

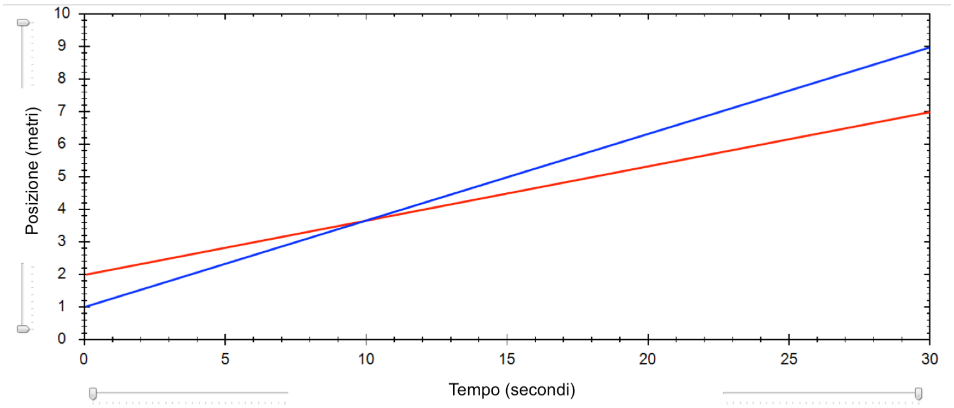
Spiegate sempre i vostri ragionamenti.

DATA ____________________ NOME ______________________________

Lavoro individuale

Scheda 4

1. Immagina che, con un altro esperimento, Rob e Bob ottengano queste linee rette:



Descrivi come si sono mossi, secondo te, Rob e Bob.
In questo esperimento, i due robottini si sono incontrati?
Spiega sempre i tuoi ragionamenti.

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 5

1. Avete già incontrato la storia della gara di corsa di Bianca e Andrea:
   Al via, Andrea parte veloce e Bianca più lenta. Poco dopo, Andrea è costretto a fermarsi per allacciarsi una scarpa. Bianca lo supera e taglia il traguardo per prima.

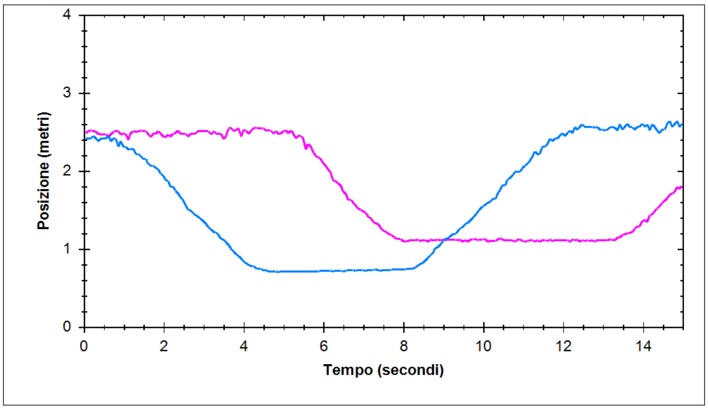
   a. Rappresentate la corsa di Bianca e Andrea utilizzando due grafici.
   b. Spiegate come avete ragionato per disegnare i grafici, aggiungendo tutte le informazioni che ritenete fondamentali per la vostra spiegazione.

2. Dove immaginate si svolga la gara? Perché? Quali informazioni sul percorso di gara forniscono i vostri grafici?

3. Disegnate il percorso di gara che avete immaginato.

Verifica secondaria I grado

1. In figura, vedi due grafici che sono stati ottenuti con WiiGraph da Giulia e Francesca durante un esperimento.

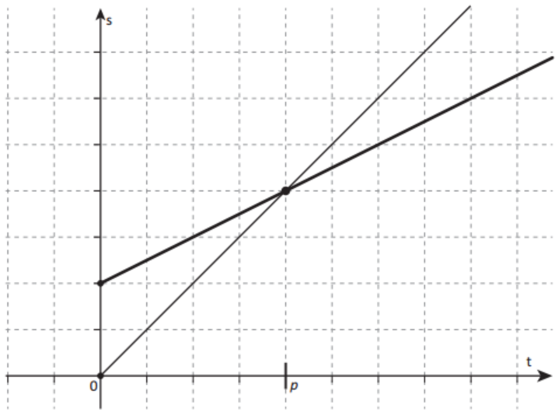


Spiega come si sono mosse, secondo te, Giulia e Francesca. Specifica tutte le informazioni che ritieni importanti per la tua spiegazione.

DATA ____________________ NOME ______________________________

Lavoro individuale

2. Considera i seguenti grafici, che rappresentano la posizione nel tempo di due treni in movimento lungo due binari paralleli.



i. Indica se le seguenti affermazioni sono vere o false, traendo le informazioni dai grafici.

A. I due treni all'istante iniziale si trovano nella stessa posizione V ☐ F ☐

B. I due treni non partono nello stesso istante V ☐ F ☐

C. I due treni all'istante iniziale hanno velocità diverse V ☐ F ☐

D. I due treni viaggiano uno a velocità doppia dell'altro V ☐ F ☐

Ora concentrati sull’istante di tempo p. Nell'istante p, i due treni:

E. si trovano nella stessa posizione V ☐ F ☐

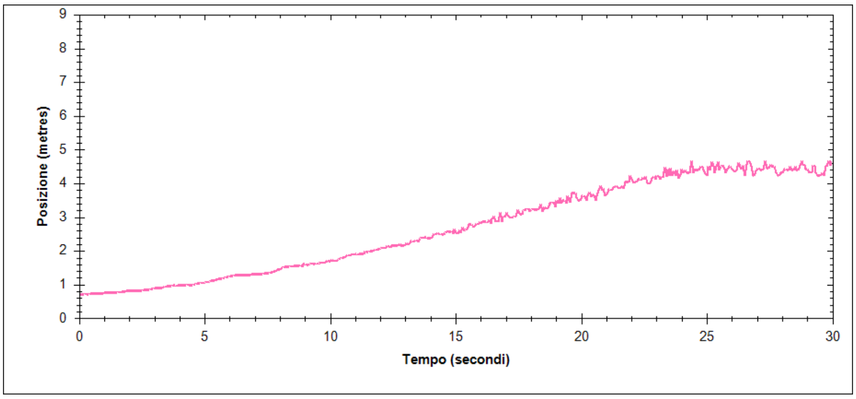
F. hanno la stessa velocità V ☐ F ☐

G. hanno percorso la stessa distanza V ☐ F ☐

H. si incontrano per una sosta V ☐ F ☐

ii. Nel caso di risposte vere, motiva la tua risposta.

3. Rob e Bob non smettono mai di fare esperimenti! Anche questa volta, vediamo solamente il grafico ottenuto da Rob con WiiGraph:



Disegna il grafico ottenuto da Bob, sapendo che:

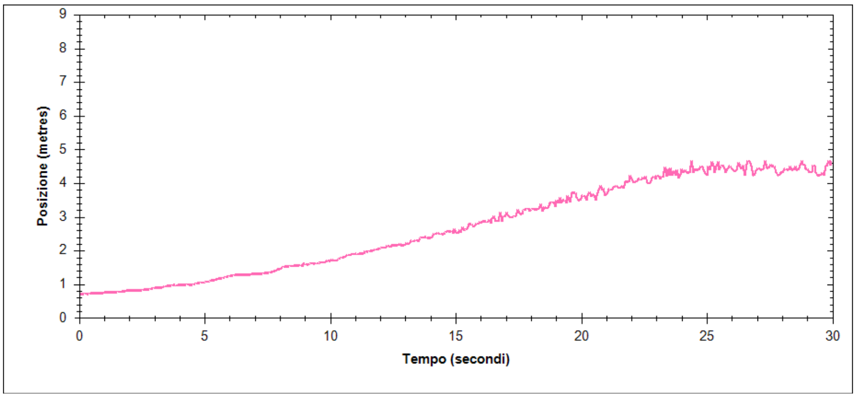
- è partito a 3 metri di distanza dal sensore;
- per i primi 15 secondi si è mosso alla stessa velocità e nello stesso verso di Rob;
- poi è rimasto fermo per 5 secondi;
- dopo essersi fermato è ripartito e, muovendosi con velocità costante, a 30 secondi è arrivato nella posizione da cui era partito.

Spiega sempre i tuoi ragionamenti.

DATA ____________________ NOME ______________________________

Lavoro individuale

4. Come cambierebbe il grafico di Bob se Bob rimanesse fermo per 10 secondi invece che per 5, tornando comunque al punto di partenza?



Disegna il grafico, motiva la tua risposta e spiega le differenze rispetto al caso precedente.

5. Spiega in che modo è possibile vedere il contributo della velocità nei grafici di posizione nel tempo. Proponi un esperimento che ti aiuti nella spiegazione e produci il grafico corrispondente (o i grafici corrispondenti!).

VERIFICA SECONDARIA I GRADO

Grade 10

Initial questionnaire

DATA ___________________ NOME ____________________________

Lavoro individuale

Questionario

Ti chiediamo di rispondere a poche e semplici domande nelle prossime pagine. Usa tutto quello (parole, disegni e altro) che ritieni necessario per essere il più chiaro possibile nelle tue risposte.
Ricorda che in questo caso non ci sono risposte giuste o sbagliate.
Vorremmo imparare a conoscerti fin da ora, con questo questionario.

1. Hai mai sentito parlare di grafici? In che contesto?
2. Spiega che cos'è per te un grafico.
3. Immagina di osservare due ragazzi che si sfidano in una gara di corsa: al via, Andrea parte veloce e Bianca più lenta. Poco dopo, Andrea è costretto a fermarsi per allacciarsi una scarpa. Bianca lo supera, e taglia il traguardo per prima.
   Se dovessi spiegare la storia che racconta la loro corsa con un diagramma, come faresti?

Questionario 1

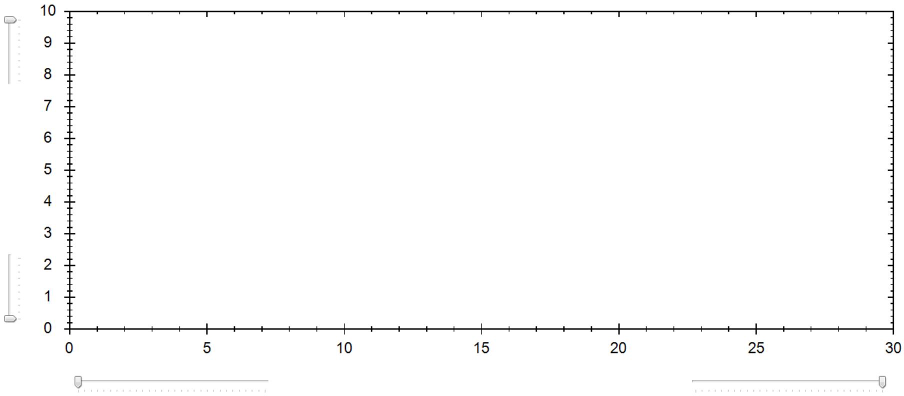
Scheda 1

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 1

1. Pensate a due grafici (non identici tra loro) e disegnateli qui sotto.



Descrivete a parole i due movimenti che, secondo voi, sono necessari per ottenere questi grafici con WiiGraph.

2. Avete a disposizione al massimo due tentativi con WiiGraph per provare a ottenere i grafici che avete disegnato. Spiegate come modifichereste la descrizione dei movimenti dopo queste prove.

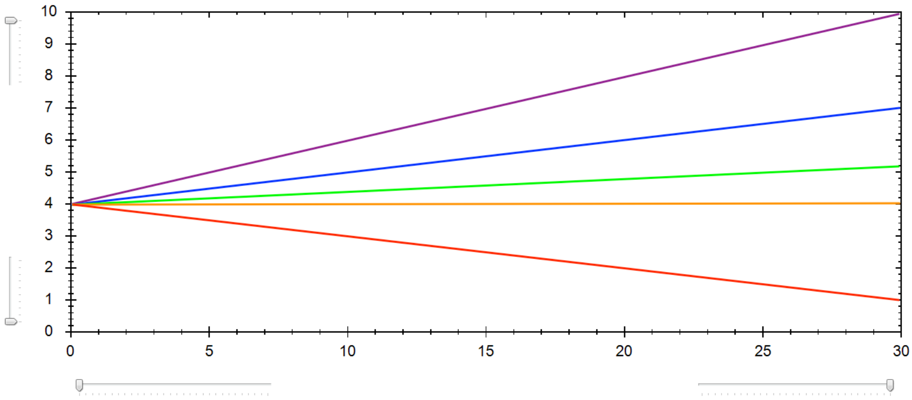
Secondaria II grado SCHEDA 1

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 2

Immaginate di poter utilizzare più di due telecomandi e di ottenere i grafici sotto:



1. Spiegate con quali movimenti si possono generare questi grafici, facendo attenzione a specificare tutte le informazioni che ritenete importanti.

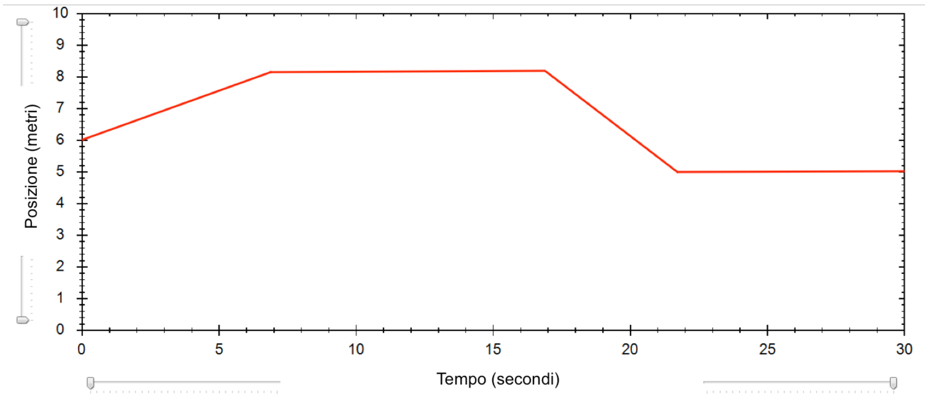
2. Secondo voi, che cosa hanno in comune le cinque rette? In che cosa invece differiscono?
   E i movimenti?
   Spiegate sempre i vostri ragionamenti.

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 3

1. Rob e Bob sono due robottini che sono stati programmati per muoversi in modo molto preciso. Insieme fanno un esperimento con i telecomandi. In corrispondenza del movimento di Rob, WiiGraph produce questo grafico:



Immaginate che Bob si sia mosso così: è partito assieme a Rob, alla stessa distanza dal sensore, ma si è mosso sempre a velocità doppia e nel verso opposto.

- Secondo voi, quale grafico mostrerebbe WiiGraph per il movimento di Bob?
- Rob e Bob, una volta partiti, si sono incontrati ancora?

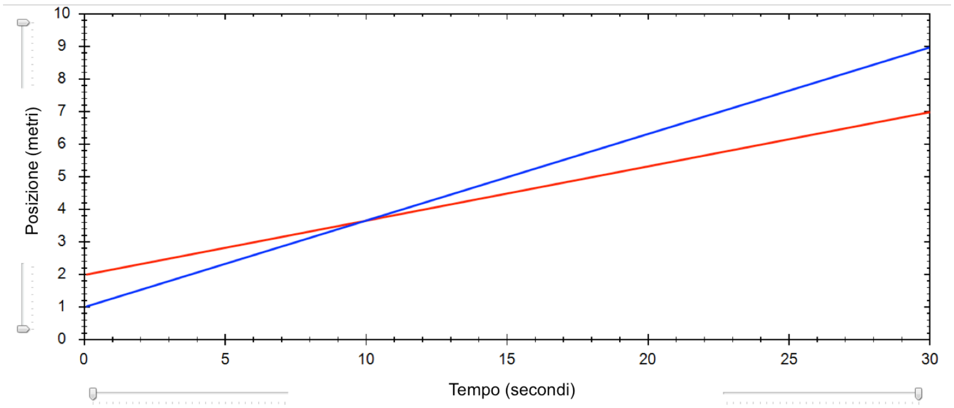
Motivate le vostre risposte.

DATA ____________________ NOME ______________________________

Lavoro individuale

Scheda 4

1. Immagina che, durante un altro esperimento, Rob e Bob producano questi grafici:



Come descriveresti il loro movimento?
I due robottini si sono incontrati questa volta?
Motiva sempre le tue risposte.

DATA ____________________ GRUPPO ______________________________

Lavoro di gruppo

Scheda 5

1. Immaginate di raccontare come funziona la somma con WiiGraph a un vostro amico che non conosce il software e non lo ha mai utilizzato. Fornitegli:
   a. una opportuna spiegazione
   b. almeno un esempio
   c. almeno una proposta di esperimento,

   aiutandovi con tutto ciò che ritenete utile e importante per essere il più chiari possibile.

2. Abbiamo già osservato che una retta orizzontale può essere ottenuta come somma di due rette orizzontali.
   Se fissiamo una particolare retta orizzontale, quali caratteristiche devono avere le rette orizzontali che, sommate, permettono di ottenerla?

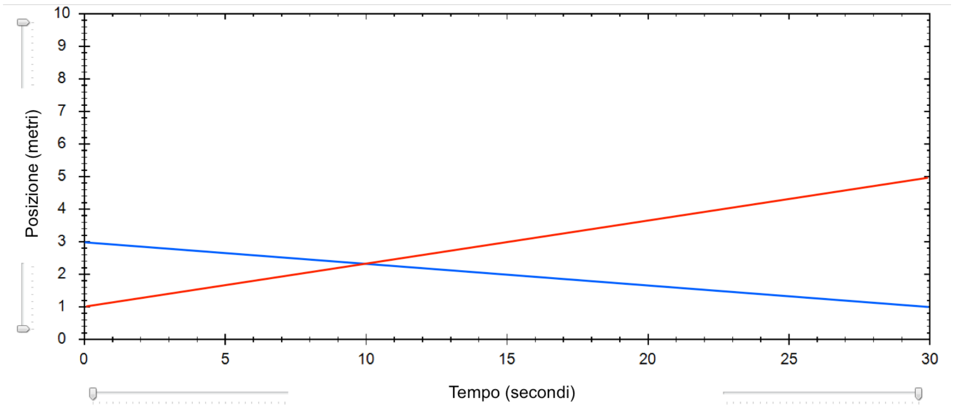
3. Giulia afferma che esiste almeno un altro modo, diverso da quello già visto, per ottenere una retta orizzontale come grafico somma. Giulia ha ragione? Perché?
   Fate anche degli esempi utili a motivare la vostra risposta.

DATA ____________________ NOME ______________________________

Lavoro individuale

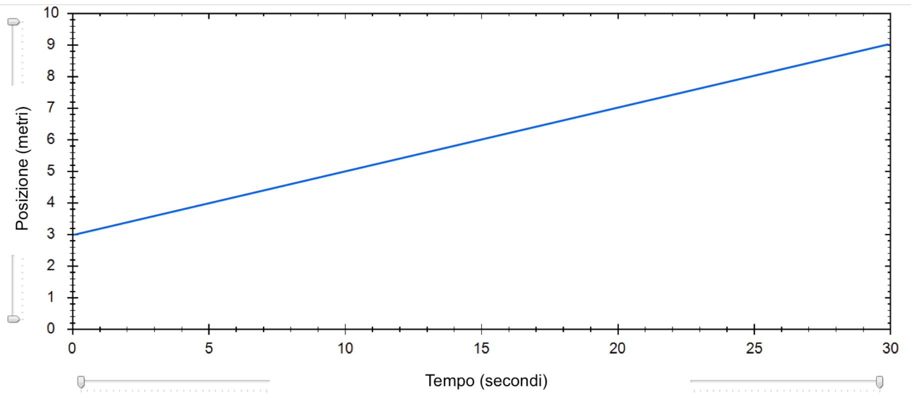
Scheda 6

1. Immagina di lavorare con la somma con WiiGraph e di produrre questi grafici con il movimento dei telecomandi. Il grafico somma non è visibile: come sarà fatto?



Disegna il grafico somma e motiva la tua risposta.

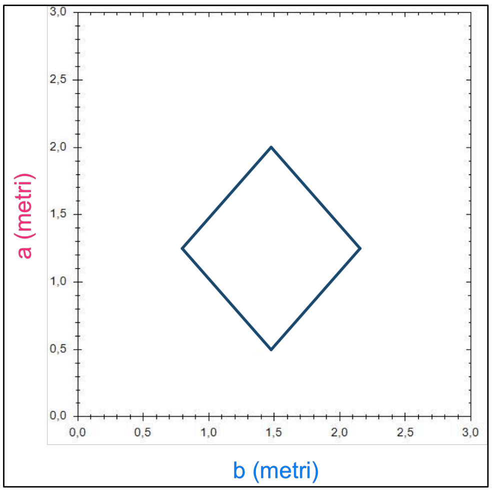
2. Immagina ora di avere come obiettivo per la somma questa retta:



Supponi di poter muovere i telecomandi solo a velocità costante. Con quali movimenti puoi ottenere questo grafico somma?
a. Descrivi i movimenti e le loro analogie e differenze.
b. Disegna i grafici corrispondenti.
c. Motiva la tua risposta.

Verifica secondaria II grado

1. Considera la figura che Giulia e Francesca hanno ottenuto con WiiGraph nell'ambiente *Versus*:



   Scegli il telecomando da associare a Giulia e quello da associare a Francesca. Spiega come si sono mosse Giulia e Francesca per creare questa figura, aiutandoti con dei diagrammi e precisando tutte le informazioni importanti.

2. Disegna i grafici di posizione nel tempo da associare ai movimenti di Giulia e di Francesca che hai considerato nell'esercizio precedente (supponi che, ad esempio, Giulia e Francesca si siano mosse esattamente per ….. secondi).

3. Durante la sperimentazione abbiamo esplorato la somma per via grafica.
   a. Come funziona, secondo te, la differenza? Proponi un esempio e un esperimento per aiutarti nella spiegazione.
   b. Immagina due grafici da cui puoi ottenere con la differenza una retta orizzontale. Come li disegneresti? Pensi che siano gli unici possibili? Perché?
   Motiva sempre le tue risposte.

VERIFICA SECONDARIA II GRADO

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